Stat205B: Probability Theory (Spring 2003) **Lecture: 20**

Embedding random variables in Brownian motion

Lecturer: James W. Pitman *Scribe: Ben Hough* jbhough@math.berkeley.edu

21 Embedding Random Variables in Brownian Motion

In this lecture, we consider a random variable X satisfying $\mathbb{E}(X) = 0$ and $\mathbb{E}(X^2) < \infty$ and show how to construct a stopping time T with $\mathbb{E}(T) < \infty$ so that $B_T \stackrel{d}{=} X$. The first step of this process is to construct a sequence X_n of simple functions converging to X a.s. In particular, we define the X_n recursively as follows:

- $X_0 = 0$, $\mathcal{G}_1 = \sigma(X > X_0)$
- $X_1 = \mathbb{E}(X|\mathcal{G}_1)$, $\mathcal{G}_2 = \sigma(\mathcal{G}_1, (X > X_1))$
- For $n > 1$, $X_n = \mathbb{E}(X|\mathcal{G}_n)$, $\mathcal{G}_{n+1} = \sigma(\mathcal{G}_n, (X > X_n))$.

It is clear that the X_n's form a martingale with respect to the $\mathcal{G}_n$'s, and we showed in the previous lecture that in fact $X_n \to X$ a.s. and in L^2. Observe also that $\mathcal{G}_n$ is a sigma field containing no more than 2^n atoms, so X_n assumes no more than 2^n values. Let A_n denote the set of possible values of X_n. That is, if $a_- = \mathbb{E}(X|X \le 0)$ and $a_+ = \mathbb{E}(X|X > 0)$:

- $A_1 = \{a_-, a_+\}$
- $A_2 = \mathbb{E}(X|X \in (-\infty, a_-]), \mathbb{E}(X|X \in (a_-, 0]), \{\mathbb{E}(X|X \in (0, a_+]), \mathbb{E}(X|X \in (a_+, \infty))\}$
- $A_3 =$ etc.

We now proceed to embed the martingale X_n in B_t. Define $T_1 = \inf\{t|B_t \in A_1\}$, and observe that $T_1 < \infty$ a.s. Since Brownian motion is a martingale w.r.t. the sigma fields $\mathcal{F}_t$ we know that $0 = \mathbb{E}(B_{\min(T_1,t)})$. Letting $t \to \infty$ it follows from the bounded convergence theorem that:

$$0 = a_-\mathbb{P}(B_{T_1} = a_-) + a_+\mathbb{P}(B_{T_1} = a_+). \tag{1}$$

On the other hand $\mathbb{E}(X_1) = 0$ so

$$0 = a_-\mathbb{P}(X_1 = a_-) + a_+\mathbb{P}(X_1 = a_+) \tag{2}$$

and therefore $X_1 \stackrel{d}{=} B_{T_1}$. Now define stopping times T_n recursively so that $T_n = \inf\{t > T_{n-1}|B_t \in A_n\}$. The strong Markov property and a computation analogous to the one just given yields:

$$(B(T_n)|B(T_1), B(T_2), \ldots, B(T_{n-1}) \stackrel{d}{=} (X_n|X_1, X_2, \ldots, X_{n-1}) \tag{3}$$

and it follows that

$$(B(T_1), B(T_2), B(T_3), \dots) \stackrel{d}{=} (X_1, X_2, X_3, \dots). \tag{4}$$

We now claim:

Theorem 21.1. The sequence of stopping times $T_n \uparrow T$, an almost surely finite stopping time with $\mathbb{E}(T) = \mathbb{E}(X^2) < \infty$ and $B(T) \stackrel{d}{=} X$.

Proof. By construction, $T_n \uparrow T$ with $T \leq \infty$. We claim that $\mathbb{E}(T_n) < \infty$. To see that this is the case, set $M_+^n = \max A_n$ and $M_-^n = \min A_n$ and note that $T_n \leq \inf\{t : B_t \notin [M_-^n, M_+^n]\}$. The latter quantity defines a stopping time whose mean is readily seen to be finite. Since $\mathbb{E}(T_n) < \infty$ we may apply Wald's identity to conclude $\mathbb{E}(T_n) = \mathbb{E}(B_{T_n}^2) = \mathbb{E}(X_n^2)$. From the previous lecture we know that $X_n \to X$ in L^2 so $\mathbb{E}(X_n^2) \to \mathbb{E}(X^2)$ and the monotone convergence theorem implies that $\mathbb{E}(T) = \mathbb{E}(X^2) < \infty$. Finally, the path continuity of Brownian motion yields $B(T_n) \xrightarrow{a.s.} B(T)$. Since $X_n \xrightarrow{a.s.} X$ with $B(T_n) \stackrel{d}{=} X_n$ it follows that $B(T) \stackrel{d}{=} X$ as desired. □

The remainder of the lecture addressed theorems (6.3) and (6.4) from Chapter 7 of Durrett, pp.404-5. See text for details.