Stat205B: Probability Theory (Spring 2003)	**Lecture: 19**

Brownian Martingales

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Reference: Section 7.5 of Durrett.

Martingales in continuous time

A process M is a martingale adapted to $(\mathcal{F}_t)$ if

1. $M_t \in \mathcal{F}_t$
2. $E(M_t|\mathcal{F}_t) = M_s$ for all $0 \leq s \leq t$.

There is a theorem which tells us, if $(\mathcal{F}_t)$ is right continuous, i.e. $\mathcal{F}_t^+ = \mathcal{F}_t$ up to null sets, then every martingale has a "version" which has right continuous paths (even with left limits). See more advanced texts, e.g. [3].

Fact: If $(\mathcal{F}_t)$ is generated by a Brownian Motion B, then every $(\mathcal{F}_t)$-Brownian Motion has a version with continuous paths. (Once the path is right continuous, it cannot have jumps). Of course, there are continuous time martingales with jumps, e.g., a compensated Poisson process $(N_t - t, t \geq 0)$, where (N_t) has stationary independent increments and N_t is Poisson with mean t.

We will be concerned with some particular martingales defined by formulae in terms of a Brownian Motion B. They will be martingales relative to the filtration $(\mathcal{F}_t)$ generated by B, that is $\mathcal{F}_t := \sigma(B_s, 0 \leq s \leq t)$, and they will obviously have continuous paths.

Examples:

1. $M_t = B_t$.
   Because B has mean 0 Gaussian distribution.
2. $M_t = B_t^2 - t$
   There is a hierarchy of polynomial martingales like this, which come from taking successive derivatives at $\theta = 0$ of the the following family of martingales:
3. $M_t^{(\theta)} = \exp(\theta B_t - \theta^2 t/2)$ for each θ real or complex.

See text for details.

Theorem 1.1 (Optional Stopping Theorem) *If (M_t) is a right continuous path martingale relative to a right contnuous filtration $(\mathcal{F}_t$, and T is a stopping time for $(\mathcal{F}_t)$ which is bounded, i.e., $T \leq C < \infty$ for some constant C, then*

$$EM_T = EM_0.$$

Proof Sketch: Use discrete approximation, and assume $T \leq C - 1$. Let $T_n = ([2^n T] + 1)/2^n$, then $T_n \downarrow T$ and by discrete time martingale theory

$$M_{T_n} = E(M_C | \mathcal{F}_{T_n})$$

Letting $n \to \infty$

$$M_T = E(M_C | \mathcal{F}_T)$$

by reversed martingale convergence. ∎

Variation or corollaries with same setup:
If T is a stopping time (no bound now), then

$$E(M_{T \wedge t} | \mathcal{F}_s) = M_{T \wedge s}$$

so $(M_{T \wedge t}, t \geq 0)$ is an $(\mathcal{F}_t)$-martingale.

To justify $EM_T = EM_0$ for unbounded $T < \infty$ a.s., we need justify the switch of limit and integral:

$$E(M_T) = E\left[\lim_{t \to \infty} M_{T \wedge t}\right] =? \lim_{t \to \infty} E(M_{T \wedge t}) = E(M_0)$$

Extra conditions justifying use of DCT or UI or L^p bounded for $p > 1$ are all that is needed. Following is a nice example:

Theorem 1.2 *If T is a stopping time of Brownian Motion B with $ET < \infty$, then*

$$E(B_T) = 0 \text{ and } E(B_T^2) = ET.$$

Proof: Note that both $(B_{T \wedge n}, n = 1, 2, \ldots)$ and $(B_{T \wedge n}, n = 1, 2, \ldots{}^2 - (T \wedge n))$ are discrete time martingales. Hence

$$E(B_{T \wedge n}) = 0 \text{ and } E(B_{T \wedge n}^2) = E(T \wedge n)$$

Let $n \to \infty$, we see $E(B_{T \wedge n}^2) \uparrow ET < \infty$. So $(B_{T \wedge n})$ is a discrete time L^2 bounded martingale which converges almost surely and in L^2 to a limit in L^2. But the almost sure limit is B_T. Therefore

$$EB_T = lim_{n \to \infty} E(B \uparrow_{T \wedge n}) = lim_{n \to \infty} 0 = 0$$

$$EB_T^2 = lim_{n \to \infty} E(B \uparrow_{T \wedge n}^2) = ET$$

∎

Embedding random variables in Brownian Motion

Theorem 1.3 *Let X be a r.v. with $EX = 0$ and $EX^2 < \infty$. Then there exists a stopping time T of Brownian Motion such that $ET < \infty$ and $B_T =_d X$. Hence (by previous theorem) $EB_T = 0$ and $EB_T^2 = ET = EX^2$*

Several different constructions of T are possible. Durett uses Skorokhod's constructions with extra randomization. The following construction due to Lester Dubins uses no extra randomization. First a little work with no Brownian Motion in view. We make a nice discrete approximations of any r.v. X on $(\Omega, \mathcal{F}, \mathcal{P})$ with $EX = 0$ and $EX^2 < \infty$.

First, let $X_0 = 0$.
Next let

$$\mathcal{G}_1 = \sigma(X > X_0), i.e., \mathcal{G}_1 = \{\emptyset, (X > X_0), (X \leq X_0), \Omega\}$$

and

$$X_1 = E(X|\mathcal{G}_1)$$

Then X_1 has two values $E(X|X > 0)$ and $E(X|X \leq 0)$ and can be rewritten as

$$X_1 = E(X|X > 0)1(X > 0) + E(X|X \leq 0)1(X \leq 0)$$

Further let

$$\mathcal{G}_2 = \sigma(\mathcal{G}_1, X > X_1)$$

$$X_2 = E(X|\mathcal{G}_2)$$

So $\mathcal{G}_2$ has 4 atoms $(X > 0, X > X_1), (X > 0, X \leq X_1), (X \leq 0, X > X_1)$ and $(X \leq 0, X \leq X_1)$ and thus X_2 typically has four possible values.
Inductively

$$\mathcal{G}_{n+1} = \sigma(\mathcal{G}_n, X > X_n)$$

$$X_{n+1} = E(X|\mathcal{G}_{n+1})$$

Then $\mathcal{G}_n$ is generated by a partition of the probability space into at most 2^n sets.

Claim 1.4 *$X_n \to X$ both almost surely and in L^2 as $n \to \infty$.*

Proof: Using $X_n = E(X|\mathcal{G}_n)$ and Jensens inequality for conditional expectation, we obtain

$$E(X_n^2) = E((E(X|\mathcal{G}_n))^2) \leq E(E(X^2|\mathcal{G}_n)) = E(X^2)$$

Hence

$$sup_n E(X_n^2) < \infty$$

So as for the martingale $(X_n, \mathcal{G}_n)$, by L^2 convergence theorem, we know

$$X_n \to X_\infty$$

both almost surely and in L^2 for some square-integrable limit X_∞. But $X_n = E(X|\mathcal{G}_n)$ implies that

$$X_n \stackrel{a.s.}{\to} E(X|\mathcal{G}_\infty)$$

where $\mathcal{G}_\infty$ is the σ-field generated by $\cup_n \mathcal{G}_n$. Thus $X_\infty = E(X|\mathcal{G}_\infty)$ and our goal is to prove $X_\infty = X$ a.s.
One proof is given by Billingsley [1]. A nicer argument is suggested by J. Neveu [2, p. 34, Exercise II-7].
Notice the following facts

$$(X > X_\infty) \subseteq \cup_n \cap_{m \geq n} (X > X_n) \subseteq (X \geq X_\infty)$$

and

$$E((X - X_\infty)1(X > X_\infty)) = E((X - X_\infty)1(X \geq X_\infty))$$

Then if let

$$G = \cup_n \cap_{m \geq n} (X > X_n)$$

Well have

$$E((X - X_\infty)1(X > X_\infty)) = E((X - X_\infty)1_G) = (E((X - X_\infty)1(X \geq X_\infty))$$

But the fact that $X_\infty = E(X|\mathcal{G}_\infty)$ makes

$$E(X1_G) = E(X_\infty 1_G)$$

since $G \in \mathcal{G}_\infty$. Hence

$$E((X - X_\infty)1(X > X_\infty)) = 0$$

Same argument on $(X < X_\infty)$ leads to

$$E((X - X_\infty)1(X < X_\infty)) = 0$$

The last two observations imply

$$E|X - X_\infty| = 0$$

We immediately obtain the desired result

$$X \stackrel{a.s.}{=} X_\infty$$

The proof of the claim is complete. ∎

References

[1] P. Billingsley. *Probability and Measure*. Wiley, New York, 1995. 3rd ed.

[2] J. Neveu. *Discrete-parameter martingales*. North-Holland Publishing Co., Amsterdam, revised edition, 1975. Translated from the French by T. P. Speed, North-Holland Mathematical Library, Vol. 10.

[3] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Springer, Berlin-Heidelberg, 1999. 3rd edition.