Stat205B: Probability Theory (Spring 2003) **Lecture: 18**

The Zero Set and Arcsine Laws of Brownian Motion

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In this lecture, we will consider the properties of the zero set of Brownian motion and introduce two arcsine laws. Throughout this lecture, let $(B_t, t \geq 0)$ be the Brownian motion starting from 0. P_x is the distribution of Brownian motion starting from x.

First of all, we define the zero set of $(B_t, t \geq 0)$ as $\mathcal{Z}(\omega) := \{t : B_t(\omega) = 0\}$. Since B has continuous paths, $\mathcal{Z}(\omega)$ is closed subset of $[0, \infty)$, which depends on ω through the path of B. Intuitively, $\mathcal{Z}(\omega)$ is a random closed subset of $[0, \infty)$. Exercise: make this rigorous by putting an appropriate σ-field on on the set of all closed subsets of $[0, \infty)$.

Now let $|\mathcal{Z}(\omega)|$ denote the Lebesgue measure of $\mathcal{Z}(\omega)$. The first result is

Theorem 18.1. $|\mathcal{Z}(\omega)| = 0$ almost surely.

Proof. For any $t \neq 0$, since $B_t \sim N(x, t)$ under P_x, we have

$$P_x(t \in \mathcal{Z}) = P_x(B_t = 0) = 0$$

i.e.

$$\int 1_{t \in \mathcal{Z}(\omega)} dP_x(\omega) \qquad \forall\ t \neq 0$$

By Fubini's theorem, we get the following relation,

$$E(|\mathcal{Z}|) = \int \int_0^\infty 1_{t \in \mathcal{Z}(\omega)} dt dP_x(\omega) = \int_0^\infty \int 1_{t \in \mathcal{Z}(\omega)} dP_x(\omega) dt = 0$$

which implies $|\mathcal{Z}| = 0$ a.s. $\square$

Another property of $\mathcal{Z}$ is

Theorem 18.2. With probability one, $\mathcal{Z}(\omega)$ has no isolated points.

Proof. Let $R_t = \inf\{u > t : B_u = 0\}$, $T_0 = \inf\{u > 0 : B_u = 0\}$. The recurrence implies $P_x(R_t < \infty) = 1$, so by the strong Markov property, we know

$$P_x(T_0 \circ \theta_{R_t} > 0 | \mathcal{F}_{R_t}) = P_0(T_0 > 0) = 0$$

Take expectation again we get

$$P_x(T_0 \circ \theta_{R_t} > 0 \text{ for some } t \in \mathbb{Q}) = 0$$

So if a point $u \in \mathcal{Z}(\omega)$ is isolated from the left, i.e. $u = R_t$ for some rational t, calculation above shows that it is an accumulated point from the right. So $\mathcal{Z}(\omega)$ has no isolated points a.s. $\square$

Now we come to see the arcsine laws for Brownian motion. There are at least three different arcsine laws. Here we will introduce two of them.

Let $T = \arg\max_{0\le t\le 1} B_t$, notice T is well defined because all the local maximum of Brownian motion are distinct. Our first arcsine law is

Theorem 18.3. For any $t \in [0,1]$, $P_0(T \le t) = \frac{2}{\pi} arcsin(\sqrt{t})$.

Proof. For any $t \in [0,1]$, let $X_r = B_{t-r} - B_t$, $Y_s = B_{t+s} - B_t$, then $(X_r, 0 \le r \le t)$ is a Brownian motion starting from 0 and is $\mathcal{F}_t$ measurable; $(Y_s, 0 \le s \le 1-t)$ is also a Browian motion starting from 0 and is independent of $\mathcal{F}_t$. With this setup, we have

$$\begin{aligned}
P_0(T \le t) &= P_0(\max_{[0,t]} B_u > \max_{[t,1]} B_u) \\
&= P_0(\max_{[0,t]}(B_u - B_t) > \max_{[t,1]}(B_u - B_t)) \\
&= P_0(\max_{[0,t]}(B_{t-r} - B_t) > \max_{[0,1-t]}(B_{t+s} - B_t)) \\
&= P_0(\max_{[0,t]} X_r > \max_{[0,1-t]} Y_s)
\end{aligned}$$

By previous work, we know $\max_{[0,t]} X_r \stackrel{d}{=} |X_t|$, $\max_{[0,1-t]} Y_s \stackrel{d}{=} |Y_{1-t}|$, and they are independent. So if we let Z_1, Z_2 are i.i.d. $\sim\ N(0,1)$, and θ is uniformly distributed on $[0, 2\pi)$, then

$$\begin{aligned}
P_0(T \le t) &= P_0(|X_t| > |Y_{1-t}|) \\
&= P(\sqrt{t}|Z_1| > \sqrt{1-t}|Z_2|) \\
&= P(\frac{|Z_2|}{\sqrt{Z_1^2 + Z_2^2}} < t) \\
&= P(|sin\theta| < \sqrt{t}) \\
&= \frac{2}{\pi} arcsin(\sqrt{t})
\end{aligned}$$

□

A little more calculation will show us the last zero point of Brownian motion on $[0,1]$ has the same distribution as T above. Let $L = \sup\{t \le 1 : B_t = 0\}$, and for $a > 0$, $T_a = \inf\{t : B_t = a\}$, by previous work, for any $t > 0$,

$$\begin{aligned}
P_0(T_a \le t) = 2P_0(B_t \ge a) &= 2\int_a^\infty (2\pi t)^{-1/2} exp(-x^2/2t)dx \\
&= 2\int_t^0 (2\pi t)^{-1/2} exp(-a^2/2s)(-t^{1/2}a/2s^{3/2})ds \\
&= \int_0^t (2\pi s^3)^{-1/2} a\ exp(-a^2/2s)ds
\end{aligned}$$

Use this formula, we have

Theorem 18.4. For any $s \in [0,1]$, $P_0(L \le s) = \frac{2}{\pi} arcsin(\sqrt{s})$.

Proof. By Markov property of Brownian motion,

$$
\begin{aligned}
P_0(L \le s) &= \int_{-\infty}^{\infty} p_s(0,x)P_x(T_0 > 1-s)dx \\
&= 2\int_0^{\infty} p_s(0,x)P_0(T_x > 1-s)dx \\
&= 2\int_0^{\infty} (2\pi s)^{-1/2} exp(-x^2/2s) \int_{1-s}^{\infty} (2\pi r^3)^{-1/2} x\ exp(-x^2/2r)drdx \\
&= \frac{1}{\pi}\int_{1-s}^{\infty} (sr^3)^{-1/2} \int_0^{\infty} x\ exp(-x^2(r+s)/2rs)dxdr \\
&= \frac{1}{\pi}\int_{1-s}^{\infty} (sr^3)^{-1/2} rs/(r+s)dr \\
&= \frac{1}{\pi}\int_0^{s} (t(1-t))^{-1/2}dt \qquad (\text{let } t = s/(r+s)) \\
&= \frac{2}{\pi} arcsin(\sqrt{s})
\end{aligned}
$$

□

R. Durrett. (1996). Probability: theory and examples.(2nd Edition) Duxbury Press.