Stat205B: Probability Theory (Spring 2003) **Lecture: 17**

Hitting Times and the Reflection Principle

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For $x \in \mathbb{R}$, let $T_x := \inf\{t : t \geq 0, B_t = x\}$. For $a, b > 0$, we wish to find $P_0(T_b < T_{-a})$ and $P_0(T_b \leq t)$.

We will solve the first problem by embedding a random walk in our Brownian Motion. The second problem will be solved by a technique called the **reflection principle**.

Claim 17.1 $P_0(T_x < \infty) = 1$ *for all* $x \in \mathbb{R}$.

Proof: This argument uses the recurrence of random walks in one dimension. Consider $B_1, B_2, B_3, \ldots$. This is a random walk with mean 0 and Gaussian increments. Thus by the Chung-Fuchs theorem it is recurrent and $P_0(B_n > x \ i.o.) = 1$. However, B_t is continuous so there are almost surely times t at which $B_t = 1$. ■

Another way to see this is by embedding of a simple symmetric random walk in Brownian motion. Let $T_\pm = T_x \wedge T_{-x}$. Then, appealing again to path continuity,

$$\begin{aligned} P_0(T_\pm \leq t) &\geq P_0(|B_t| > x) \\ &= P_0(|B_1| > \frac{x}{\sqrt{t}}) \longrightarrow 1 \text{ as } t \longrightarrow \infty \end{aligned}$$

In fact we can get $P_0(T_\pm > t) \leq Ce^{-\theta t}$ for some $C > 0$ and $\theta > 0$. Finally,

$$P_0(T_\pm = T_x) = P(B(T_\pm) = x) = P(B(T_\pm) = -x) = \frac{1}{2}$$

by symmetry.

Now fix $\delta > 0$ and consider the succesive times when B_t hits the lattice $\delta\mathbb{Z}$. Let $S_0 = 0$ and

$$S_{n+1} = S_n + T_{\pm\delta}(B^{(n)}),$$

where

$$B^{(n)}(t) = B(S_n + t) - B(S_n).$$

Each S_n is a stopping time of B_t and each $B_t^{(n)}$ is a Brownian Motion independent of $(B_t, 0 \leq t \leq S_n)$. By repeated application of the Strong Markov Property to justify the independce and symmetry to get $p = q = \frac{1}{2}$ we have that $(\frac{B(S_n)}{\delta}, n = 1, 2, \ldots)$ is a simple symmetric random walk. (We can also embed many other walks and processes such as martingales in a Brownian Motion.)

Now observe that by the solution of the gambler's ruin problem for simple symmetric random walk,

$$\begin{aligned} P_0(T_{2\delta} < T_{-\delta}) &= P_0(\frac{B(S_n)}{\delta} \text{ hits 2 before } -1) = \frac{1}{3} \\ P_0(T_{k\delta} < T_{-m\delta}) &= \frac{m\delta}{k\delta + m\delta} \\ P_0(T_a < T_{-b}) &= \frac{b}{a+b} \end{aligned}$$

if $a = k\delta,\ b = m\delta$ for some $\delta > 0$ (ie. if $\frac{a}{b}$ is rational). For fixed b, $P_0(T_a < T_{-b})$ decreases as a increases. Thus by the continuity of the right hand side of the equation we conclude that the formula holds for all positive real a and b.

As a check, the fact that $P_0(T_x < \infty) = 1$ for all $x \in \mathbb{R}$ follows easily from the formula $P_0(T_a < T_{-b}) = b/(a+b)$. Now we wish to find the distribution of T_x for $x > 0$. Let

$$M_t := \max_{0 \le s \le t} B_s$$

and notice that $(T_x, x \ge 0)$ is the left continuous inverse for $(M_t, t \ge 0)$. Let

$$T_+ := \inf\{t : B_t > 0\},\ T_- := \inf\{t : B_t < 0\}$$

Claim 17.2 *$T_+ = T_- = 0$ a.s.*

Proof: $P(\forall \epsilon > 0\ B_t > 0 \text{ for some } t \le \epsilon) > \frac{1}{2}$ and $\{\forall \epsilon > 0\ B_t > 0 \text{ for some } t \le \epsilon\} \in \mathcal{F}_0^+$ Thus by Bluementhals 0/1 Law, $P(T_+ = 0) = 1$. ■

Claim 17.3 *With probability one, B_t cannot reach and not exceed the same value more than twice.*

Proof: For each fixed $a > 0$, consider the first hit of a local max after staying below the max for at least time a. This is a stopping time, so by the Strong Markov Property and the previous result, the process exceeds the value immediately after hitting it for the second time. This argument can be repeated to consider the nth time a local max is hit after staying below the max for at least time a. Finally, let $a \downarrow 0$ to finish the argument. ■

The Reflection Principle $(M_t \ge x) = (T_x \le t)$ so if we know the distribution of M_t for all $t > 0$ then we know the distribution of T_x for all $x > 0$. Define the reflected path,

$$\hat{B}(t) = \begin{cases} B(t) & \text{if } t \le T_x \\ x - (B(t) - x) & \text{if } t > T_x \end{cases}$$

By an application of the Strong Markov Property and because B and $-B$ are equal in distribution we can deduce that $\hat{B}$ and B are equal in distribution. (Rigorous proof of this involves some measurability issues which are not entirely trivial: see e.g. Freedman's *Brownian Motion and Diffusion* or Durrett's text for details)

Observe that for $x, y > 0$,

$$(M_t \ge x, B_t \le x - y) = (\hat{B}_t \ge x + y)$$

so

$$P_0(M_t \ge x, B_t \le x - y) = P_0(B_t \ge x + y)$$

Taking $y = 0$ in the previous expression we have

$$P_0(M_t \ge x, B_t \le x) = P_0(B_t \ge x)$$

$(B_t > x) \subset (M_t \ge x)$, so

$$P_0(M_t \ge x, B_t > x) = P_0(B_t > x) = P_0(B_t \ge x)$$

by continuity of the distribution. Adding these two results we find that

$$P_0(M_t \ge x) = 2P_0(B_t \ge x)$$

In words, the distributions of M_t and $|B_t|$ are the same.

Now recall that $P_0(M_t \geq x) = P_0(T_x \leq t)$ so

$$\begin{aligned} P_0(T_x \leq t) &= P_0(|B_t| \geq x) = P_0(\sqrt{t}|B_1| \geq x) \\ &= P_0(B_1^2 \geq \frac{x^2}{t}) = P_0(\frac{x^2}{B_1^2} \leq t) \end{aligned}$$

So T_x is equal in distribution to $\frac{x^2}{B_1^2}$. As a check, T_x has the same distribution as $x^2 T_1$, which is explained by Brownian scaling.