Stat205B: Probability Theory (Spring 2003) **Lecture: 16**

The Strong Markov Property

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Idea: (Itô-Mckean) The Brownian traveller starts afresh at stopping times.

Definition. A *Filtration* is an increasing family of σ-fields $(\mathcal{F}_t, t \in \mathcal{I})$ where $\mathcal{I} \subseteq \Re$ is some index set. We are familiar with the case when $\mathcal{I} = \{0, 1, 2, ...\}$, now we extend it to $\mathcal{I} = [0, \infty)$.

We say $T : \Omega \to \mathcal{I} \cup \{\infty\}$ is a stopping time if $(T \leq t) \in \mathcal{F}_t$ for all $t \in \mathcal{I}$. Intuitively, we can think of $\mathcal{F}_t$ as the information available up to time t.

Remark. Comparing to Durrett (Section 7.3, pp 387-389), we use $(T \leq t)$ instead of $(T < t)$. The reasons are the following:

- This is consistent with the definition of stopping time in discrete time.
- We replace $(\mathcal{F}_t)$ by $(\mathcal{F}_t^+)$ in continuous time $[0, \infty)$ with $\mathcal{F}_t^+ := \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$, therefore $(\mathcal{F}_t^+)$ is a new filtration and T satisfies:

$$(T < t) \in \mathcal{F}_t \text{ for all t} \Longleftrightarrow (T \leq t) \in \mathcal{F}_t^+ \text{ for all t}$$

If T is a stopping time relative to $\mathcal{F}_t$, then

$$(T = t) \in \mathcal{F}_t \text{ for all t}$$

In discrete time, this is equivalent to T being a stopping time, but not continuous time since

$$(T \leq t) = \cup_{(0 \leq s \leq t)} (T = s)$$

is an uncountable union.

Example. Suppose we start with a filtration $(\mathcal{F}_t)$ and complete it with $(\bar{\mathcal{F}}_t) = \sigma(\mathcal{F}_t, \mathcal{N})$ where $\mathcal{N}$ is the collection of events of probability 0. Let T be any random variable with a continuous distribution, then

$$P(T = t) = 0 \text{ for all } t \text{ , } (T = t) \in \mathcal{N} \text{ and } (T = t) \subseteq \mathcal{F}_t$$

so T is an $(\bar{\mathcal{F}}_t)$-stopping time if we use this definition.

Definition. If T is an $(\mathcal{F}_t)$-stopping time, then $\mathcal{F}_T$, the *σ-field of events determined by time T*, is defined as:

$$\mathcal{F}_T := \{A \in F : A \cap (T \leq t) \in \mathcal{F}_t\}$$

Easily we can check the following:

- $\mathcal{F}_T$ is a σ-field.

- If $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$ for some process X with continuous path, then things like T, X_T, $X_{T\wedge t}$ which can be considered as being constructed from $(X_s, 0 \leq s \leq T)$ are all $\mathcal{F}_T$-measurable.

Roughly, $\mathcal{F}_T = \sigma(X_s, 0 \leq s \leq t)$, and following are sensible facts about $\mathcal{F}_T$:

- If $S \leq T$ are two stopping times, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.
- If $T_n \downarrow T$ and T_ns are $(\mathcal{F}_t^+)$-stopping times, then T is an $(\mathcal{F}_t^+)$-stopping time and $\mathcal{F}_T^+ = \cap_n \mathcal{F}_{T_n}^+$.

Note that for $(\mathcal{F}_t)$ generated by Brownian motion the difference between $(\mathcal{F}_t)$ and $(\mathcal{F}_t^+)$ is unimportant: it turns out that $\mathcal{F}_t = \mathcal{F}_t^+$ a.s., meaning that if for all $A \in \mathcal{F}_t^+$, there exists $B \in \mathcal{F}_t$ such that $1_A = 1_B$ a.s.. The case $t = 0$ of this assertion is Blumenthal's zero-one law, stated below.

Example. Let $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$ for X with continuous paths, $T = \inf\{t : X_t = 1\}$. This is an $(\mathcal{F}_t)$-stopping time. This can ve checked as follows. Start from

$$(T \leq t) = (X_s = 1 : \text{ for some } 0 \leq s \leq t).$$

Let D be a countable dense subset of $[0, \infty)$. By considering X_s for $s \in D$,

$$(T \leq t) = \cap_{n=1}^{\infty} \cup_{(s\in D, 0\leq s\leq t)} \{|X_s - 1| \leq \frac{1}{n}\}$$

However, if we let

$$T = \inf\{t : X_t > 1\}$$

then T is not an $(\mathcal{F}_t)$-stopping time but an $(\mathcal{F}_t^+)$-stopping time since

$$(T \leq t) = \cap_{\epsilon>0}\{X_s > 1 \text{ for some } 0 \leq s \leq t + \epsilon\} \in \mathcal{F}_t^+$$

Theorem 16.1. (Strong Markov property of Brownian Motion) If B is an $(\mathcal{F}_t)$-Brownian Motion and T is an $(\mathcal{F}_t^+)$-stopping time, then given $(T < \infty)$, $(B_{T+S} - B_T, S \geq 0)$ is a Brownian Motion which is independent of $(\mathcal{F}_T^+)$.

Remark. We say B is an $(\mathcal{F}_t)$-Brownian Motion means $B_t \in \mathcal{F}_t$, $B_0 = 0$, B has continuous paths and for every s and t, $B_{t+s} - B_t \sim N(0, s)$, and $B_{t+s} - B_t$ is independent of $\mathcal{F}_t$. If B is a Brownian Motion in previous sense then it is an $(\mathcal{F}_t)$-Brownian Motion for $(\mathcal{F}_t)$ the filtration generated by B.

Proof. We break the proof into three steps.

Step 1: Take T be a fixed (non-random) time. We know from previous lecture (from Gaussian FDDs) that the statement is true with independence of $\mathcal{F}_T$ instead of $\mathcal{F}_{T+}$.

Step 2: Take T to be an $(\mathcal{F}_t)$-stopping time with discrete distribution

$$T = \Sigma_{n=1}^{\infty} t_n \cdot 1_{(T=t_n)} \text{ for some } 0 < t_1 < t_2 < ...$$

We can get conclusion with independence of $\mathcal{F}_T$ by conditioning on $T = t_n$, then use result for fixed time t_n and sum over n.

Step 3: Take T to be a general $(\mathcal{F}_t^+)$-stopping time. Let

$$T_k = \text{ the least multiple of } 2^{-k} \text{ that is } > T.$$

Note that $\mathcal{F}_T^+ \subseteq \mathcal{F}_{T_k}$.

By Step 2, we know given $(T < \infty)$, $(B_{T_k+s} - B_{T_k}, s \geq 0)$ is Brownian Motion independent of $\mathcal{F}_{T_k}$ and hence independent of $\mathcal{F}_T^+$. Letting $k \to \infty$, $T_k \downarrow T$ on $(T < \infty)$, we want to conclude the following:

$$(B_{T+s} - B_T, s \geq 0) \text{ is Brownian Motion independent of } \mathcal{F}_T.$$

Brownian Motion requires appropriate FDDs and path continunity (which can be easily verified). Therefore we just need to check FDDs. We shall do this just in the one-dimensional case, i.e. for each fixed $s > 0$

$$B_{T+s} - B_T \sim N(0, s) \text{ and is independent of } \mathcal{F}_T$$

We know $(B_{T_k+S} - B_{T_k}, S \geq 0) \sim N(0, s)$ and are independent of F_T and $T_k \downarrow T$, hence

$$B_{T_k+S} \to B_{T+S} \text{ and } B_{T_k} \to B_T \text{ as } k \to \infty$$

With a little measure theory, we reduce to showing for all bounded and continuous functions f and all $A \in \mathcal{F}_T^+$,

$$E[f(B_{T+s} - B_T) \cdot 1_A] = E[f(B_s)] \cdot P(A) \tag{1}$$

(This reduction uses the fact that the distribution a random variable X is determined by $Ef(X)$ for f bounded and continuous). We know (1) is true if we replace T with T_k in (1) with f bounded and continuous. Therefore by BCT, (1) holds for all f bounded and continuous by passage to limit in (1) with $T_k \to T$. So the result follows. $\square$

As a corollary of the above, we have the following.

Theorem 16.2. (Blumenthal 0-1 Law) Take $T = 0$ in above, and let $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$, then $(B_s, s \geq 0)$ is a BM independent of $\mathcal{F}_0^+$. However, since $\mathcal{F}_0^+ \subseteq \sigma(B_s, s \geq 0)$, therefore $\mathcal{F}_0^+$ is independent of itself, consequently $P(A) = 0$ or 1 for all $A \in \mathcal{F}_0^+$.