Basic Properties of Brownian Motion

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In this lecture, we discuss some basic properties of Brownian motion, including various transformations, the transition semigroup and its generator.

Brownian motion lies in the intersection of several important classes of processes. It is a Gaussian Markov process, it has continuous paths, it is a process with stationary independent increments (a Lévy process), and it is a martingale. Several characterizations are known based on these properties.

We consider also the following variation of Brownian motion:

Example 15.1. Given a Brownian motion $(B_t, t \geq 0)$ starting from 0. Let $X_t = x+\delta t+\sigma B_t$, then $(X_t, t \geq 0)$ is a Gaussian processes, i.e. all its FDDs (finite dimensional distributions) are multivariate normal. Note that X is a Markov process, with stationary independent increments, with x the initial state, δ the drift parameter, σ^2 the variance parameter. These three parameters determine all the FDDs of $(X_t, t \geq 0)$, which may be called a Brownian motion started at x with drift parameter δ and variance parameter σ^2.

Note that the FDDs of a Gaussian process $(X_t, t \in I)$ are uniquely determined by the mean function $t \to E(X_t)$ and the covariance function $(s,t) \to Cov(X_s X_t)$. Notice the covariance function must be symmetric and non-negative definite.

With this idea, we have a second equivalent definition of Brownian motion which is useful:

Definition 15.2. A real valued process $(B_t, t \geq 0)$ is a Brownian motion starting from 0 iff
(a) (B_t) is a Gaussian process;
(b) $EB_t = 0$ and $EB_sB_t = s \wedge t$, for all $s, t \geq 0$;
(c) With probability one, $t \to B_t$ is continuous.

This definition is often useful in checking that a process is a Brownian motion, as in the transformations described by the following examples based on $(B_t, t \geq 0)$ a Brownian motion starting from 0.

Example 15.3 (scaling). For each $s > 0$, $(s^{-1/2}B_{st}, t \geq 0)$ is a Brownian motion starting from 0. This is easy to verify if we use the definition above. Moreover, for each $s > 0$, $(B_{st}, t \geq 0) \stackrel{d}{=} (s^{1/2}B_t, t \geq 0)$, i.e. all the FDDs are the same.

Example 15.4 (shifting). For each $s > 0$, $(B_{s+t} - B_s, t \geq 0)$ is a Brownian motion starting from 0, and this Brownian motion is independent of $(B_u, 0 \leq u \leq s)$. This form of the Markov property of Brownian motion of B follows easily from the stationary independent increments of B.

Example 15.5 (time reversal). Consider $(B_t, 0 \leq t \leq 1)$, define $(X_t, 0 \leq t \leq 1)$ by $X_t = B_{1-t} - B_1$, then $(X_t, 0 \leq t \leq 1) \stackrel{d}{=} (B_t, 0 \leq t \leq 1)$.

Example 15.6 (inversion). The process $(X_t, t \geq 0)$ defined by $X_0 = 0$ and $X_t = tB(1/t)$ for $t > 0$ is a Brownian motion starting from 0. To see this, notice (a) $(X_t, t \geq 0)$ is a Gaussian process; (b)$E(X_t) = 0$, and if $s < t$ then

$$E(X_sX_t) = stE(B(1/s)B(1/t)) = s$$

(c)X clearly has paths that are continuous in t provided $t > 0$. To handle $t = 0$, we note X has the same FDD on a dense set as a Brownian motion starting from 0, then recall in the previous work, the construction of Brownian motion gives us a unique extension of such a process, which is continuous at $t = 0$. An alternative method is the following:

Direct proof of continuity. By strong law of large numbers, $B_n/n \to 0$ as $n \to \infty$ through the integers. To handle values between integers, use Kolmogorovs inequality,

$$P(\sup_{0<k\leq 2^m} |B(n + k2^{-m}) - B_n| > n^{2/3}) \leq n^{-4/3} E(B_{n+1} - B_n)^2$$

Let $m \to \infty$ we get

$$P(\sup_{u\in[n,n+1]} |B_u - B_n| > n^{2/3}) \leq n^{-4/3}$$

Since $\sum_n n^{-4/3} < \infty$, the Borel-Cantelli lemma implies $B_u/u \to 0$ as $u \to \infty$. Taking $u = 1/t$, we have $X_t \to 0$ as $t \to 0$.

Now, let $\mathbb{P}_0$ be the Wiener measure i.e. the distribution of $(B_t, t \geq 0)$ starting from $B_0 = 0$, and let $\mathbb{P}_x$ be the distribution of $(x + B_t, t \geq 0)$. Consider the Brownian motion as a time-homogenous Markov process with transition kernel

$$p_t(x, dy) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy$$

Generally, given a group of probability kernels $\{p_t, t \geq 0\}$, we can define the corresponding **transition operators** as $P_t f(x) := \int p_t(x, dy) f(y)$ acting on bounded or non-negative measurable functions f. There is an important relation between these two things:

Theorem 15.7 (semigroup property). The probability kernels $\{p_t, t \geq 0\}$, satisfy the Chapman-Kolmogrov relation iff the corresponding transition operators $\{P_t\}$ have the semigroup property

$$P_{s+t} = P_s \circ P_t \qquad s, t \geq 0$$

Proof. For each bounded and measurable f,

$$\begin{aligned} P_{t+s} f(x) &= \int p_{s+t}(x, dy) f(y) \\ &= \int p_s(x, dz) \int p_t(z, dy) f(y) \qquad \text{Chapman-Kolmogrov relation} \\ &= \int p_s(x, dz) P_t(f(z)) = (P_s \circ P_t) f(x) \end{aligned}$$

□

If $\{p_t, t \geq 0\}$ is the family of transition kernels of a Markov process, the Markov property guarantees the Chapman-Kolmogrov relation, so the family of opertators $\{P_t\}$ associated with a Markov process is a semigroup. The **generator** $\mathcal{Q}$ of the transition semigroup $\{P_t\}$ is an operator defined as

$$\mathcal{Q} f := \lim_{t\downarrow 0} \frac{P_t f - f}{t}$$

for suitable f, meaning that the limit exists in some sense.

Now consider the semigroup of transition operators $\{P_t\}$ and its generator for Browning motion. By definition,

$$\begin{aligned} P_t f(x) &= \int_{\mathbb{R}} p_t(x,y) f(y) dy \\ &= \int \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} f(y) dy \\ &= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} f(x+\sqrt{t}z) dz \end{aligned}$$

As for the generator, for f with two continuous derivatives f' and f'' such that f'' is bounded,

$$\begin{aligned} \mathcal{Q} f(x) &= \lim_{t \downarrow 0} \frac{P_t f(x) - f(x)}{t} \\ &= \lim_{t \downarrow 0} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{f(x+\sqrt{t}z) - f(x)}{t} dz \\ &= \lim_{t \downarrow 0} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{f'(x)\sqrt{t}z + f''(x+\theta\sqrt{t}z) t z^2/2}{t} dz \\ &= \lim_{t \downarrow 0} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{f''(x+\theta\sqrt{t}z) z^2}{2} dz \\ &= \frac{1}{2} f''(x) \end{aligned}$$

where $\theta \in [0,1]$ is function of x and $\sqrt{t}z$, so as $t \downarrow 0$ there is the convergence $\theta\sqrt{t}z \to 0$, hence $f''(x+\theta\sqrt{t}z) \to f''(x)$ by continuity of f'', and the last step is justified by the dominated convergence theorem, using the assumption that f'' is bounded.

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