Stat205B: Probability Theory (Spring 2003) **Lecture: 13-14**

Construction of Brownian Motion

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The physical phenomenon of Brownian motion was discovered by a 19th century scientist named Brown, who observed through a microscope the random swarming motion of pollen grains in water, now understood to be due to molecular bombardment. The theory of Brownian motion was developed by Bachelier in his 1900 PhD Thesis *Théorie de la Spéculation*, and independently by Einstein in his 1905 paper which used Brownian motion to estimate the size of molecules. The modern treatment of Brownian motion (BM), also called the *Wiener process* is due to Wiener in 1920s. Wiener proved that there exists a version of BM with continuous paths. Lévy also made major contributions to the theory. Note that BM is a martingale, a Markov process, and a Gaussian process. Hence its importance in the theory of stochastic process. It serves as a building block for many more complicated processes.

Definition. Let $(\Omega, \mathcal{F}, P)$ be a probability space. $(X(t,\omega), t \geq 0, \omega \in \Omega)$ is a Browian motion if

1. For each t, $X_t = X(t, \cdot)$ is a random variable.
2. For each ω, $t \to X(t, \omega)$ is continuous.
3. The distribution of X_t is normal with mean 0 and variance t.
4. The process has stationary independent increments, i.e. if $0 \leq t_1 < t_2 < ... < t_n$, then $X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent, and the distribution of $X_{t_i} - X_{t_{i-1}}$ is normal with mean 0 and variance $t_i - t_{i-1}$.

Because of the convolution properties of normal distributions, the joint distribution of $X_{t_1}, ..., X_{t_n}$ are consistent for any $t_1 < ... < t_n$. By Kolmogorov's consistency theorem, given such a consistent family of finite dimensional distributions, there exists a process $(X_t, t \geq 0)$ satisfying 1), 3), 4). This construction gives $\Omega = \mathbb{R}^{[0,\infty)}$ and $\mathcal{F} = B^{[0,\infty)}$. But every event in $\mathcal{F}$ depends on only countably many indices, while continuity depends on all indices t such that $0 \leq t < \infty$. Thus, $\{\omega : X(t,\omega)$ is continuous in t, for all $t\}$ is not $\mathcal{F}$-measurable, and completion doesn't help. So some care is required to achieve the continuity property (2).

Claim: Browian motion exists.

Proof: Following Freedman [1], we will proceed by the following steps.

- Step 1: Construct X_t for $t \in D$ = {diadic rationals} = $\{\frac{k}{2^n}\}$.
- Step 2: Show for almost all ω, $t \to X(t,\omega)$ is uniformly continuous on $D \cap [0, T]$ for any finite T.
- Step 3: For such ω, extend to whole $[0, \infty)$ by continuity.
- Step 4: Check that (1), (2) and (4) still hold for the process so defined.

Checking Step 1. Conditions (1) and (4) give consistent set of finite dimensional distribution for $\{X_t, t \in D\}$. So there exists a process $\{X_t, t \in D\}$ satisfying conditions (1), (3) and (4) by Kolmogrov consistency theorem. Or , we can explicitly construct $\{X_t, t \in D\}$ by interpolation procedure from a sequence $Z_1, Z_2, ...$ of iid $N(0, 1)$, using the normality of the conditional distribution of $X_{\frac{1}{4}}$ given $X_0 \& X_{\frac{1}{2}}$, e.g. let $X_1 = Z_1, X_0 = 0$,

$$\begin{aligned} X_{\frac{1}{2}} &= \tfrac{1}{2}(X_0 + X_1) + \tfrac{1}{2}Z_1 \\ X_{\frac{1}{4}} &= \tfrac{1}{2}(X_0 + X_{\frac{1}{2}}) + 2^{-\frac{3}{2}}Z_2 \\ X_{\frac{3}{4}} &= \tfrac{1}{2}(X_{\frac{1}{2}} + X_1) + 2^{-\frac{3}{2}}Z_3 \end{aligned}$$

and so on. Check that the finite dimensional distributions are right and, and extend to $(X_t, t \in D)$ by independent repetitions of the construction on $[0, 1]$ to get the increments on $[1, 2]$, $[2, 3]$,... .

Checking step 2. We will show for almost all ω there is a $K(\omega)$ such that $k \geq K(\omega)$ and $t \in D \cap [0, 1]$ imply

$$|X(t, \omega) - X(t_k, \omega)| \leq b_k$$

where t_k = largest multiple of $2^{-k} \leq t$, and $b_k \to 0$ as $k- > \infty$. Recall that if $|f(t) - f(t_m)| \leq b_k$ for all $t \in [0, 1] \cap D$, then $|f(t) - f(s)| \leq 2b_k$, provided $|s - t| \leq 2^{-(k+1)}$. Now,

$$\begin{aligned} & P(|X(t, \omega) - X(t_k, \omega)| > b_k, \text{for some t} \in [0, 1] \cap D) \\ \leq \; & 2^k P(|X_t| > b_k, \text{for some t} \in D, 0 \leq t \leq 2^{-k}) \\ = \; & \lim_{n\to\infty} 2^k P(|X_t| > b_k, \text{for some t} = \tfrac{j}{2^n}, j = 0, 1, 2, ..., 2^{n-k}) \end{aligned}$$

Note : Kolmogrov maximum inequality does not help here because we have to multiply 2^{k+1} .

Here , by using symmetry and Lévy inequality,

$$\begin{aligned} & \lim_{n\to\infty} 2^k P(|X_t| > b_k, \text{for some t} = \tfrac{j}{2^n}, j = 0, 1, 2, ..., 2^{n-k}) \\ \leq \; & 2^{k+1} P(|X_{\frac{1}{2^{-k}}}| > b_k) \quad \text{Lévy inequality} \\ = \; & 2^{k+2} P(Z > b_k 2^{\frac{k}{2}}) \quad \text{where} \quad Z \sim N(0, 1) \\ = \; & 2^{k+2} \int_{b_k 2^{\frac{k}{2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \quad \text{recall} \quad P(Z > h) \leq \frac{\phi(h)}{h} \\ \leq \; & \frac{2^{k+2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} b_k^2 2^k}}{b_k 2^{\frac{k}{2}}} \end{aligned}$$

So finally want to show that we can chose b_k in such a way that

1. $b_k \to 0$
2. $\sum_k 2^k \frac{e^{-\frac{1}{2} b_k^2 2^k}}{b_k 2^{\frac{k}{2}}} < \infty$

It is easy to check that $b_k = k^{-1}$ satisfies these requirements. Thus, by Borel-Cantelli lemma

$$P(|X(t, \omega) - X(t_k, \omega)| > b_k, i.o.) = 0$$

Hence, $t \to X(t, \omega)$ is uniformly continuous.

Checking step 3. By the step 2, for almost all ω, $X(t, \omega)$admits a continuous extension from $t \in D$ to $t \in [0, \infty)$.

Checking step 4. The continuity property (2) is clear by construction, and the finite dimensional distribution come out right for all real times by obvious limit arguments.

References

[1] D. Freedman. *Brownian motion and diffusion*. Springer-Verlag, New York, second edition, 1983.