Stat205B: Probability Theory (Spring 2003) **Lecture: 11**

Range of Random Walks

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Two topics will be covered in this lecture:

- Application of the Ergodic Theorem to range of random walks (Kesten-Spitzer-Whitman Theorem, see Durett section 6.3).
- A variation of the proof of the Ergodic Theorem without using the maximal inequality $E(X \cdot 1_{(M_n>0)}) \geq 0$.

The range of a random walk Consider a random walk on $\mathbb{Z}^d$ for some $d \geq 1$ or more generally on a discrete group with operation '+'). Let $S_n := X_1 + X_2 + ... + X_n$ where the X_i are *i.i.d.* taking values in the group. (Or see text for a formulation with the X_i just assumed stationary). The *range* of the walk up to time n is the number of distinct states visited by time n, that is

$$R_n := \mid \{0, S_1, S_2, ..., S_n\} \mid .$$

How does R_n behave as $n \to \infty$?

Theorem 11.1. **(Kesten-Spitzer-Whitman)**

$$\frac{R_n}{n} \to P(S_1 \neq 0,\, S_2 \neq 0, ...)\, a.s.$$

Remark: According to Markov chain theory the above limit is 0 or > 0 according to whether the walk is recurrent or transient.

Proof. We will use method of indicators. Let

$$\begin{aligned} R_n &= 1 + 1_{(S_1 \neq 0)} + 1_{(S_2 \neq 0,\, S_2 \neq S_1)} + ... + 1_{(S_n \neq 0,\, S_n \neq S_1,\, S_n \neq S_2, ...,\, S_n \neq S_{n-1})} \\ &= 1 + 1_{(X_1 \neq 0)} + 1_{(X_1 + X_2 \neq 0,\, X_2 \neq 0)} + ... + 1_{(X_1 + ... + X_n \neq 0, X_2 + ... + X_n \neq 0, ..., X_n \neq 0)} \\ &\leq 1 + 1_{(X_1 \neq 0)} + 1_{(X_2 \neq 0)} + ... + 1_{(X_n \neq 0)} \end{aligned}$$

This implies that

$$\frac{R_n}{n} \leq \frac{1}{n} + (\sum_{i=1}^{n} 1_{(X_i \neq 0)})/n$$

By strong law of large numbers for $I.I.D.$ case or ergodic theorem for stationary ergodic case, we have

$$(\sum_{i=1}^{n} 1_{(X_i \neq 0)})/n \to P(X_1 \neq 0)\, a.s.$$

Therefore we can conclude that

$$\limsup_{n\to\infty} \frac{R_n}{n} \le P(X_1 \neq 0) = P(S_1 \neq 0)\, a.s.$$

Similarly we can get

$$\frac{R_n}{n} \le \frac{2}{n} + \frac{1}{n}\{1_{(X_1+X_2\neq 0,\, X_2\neq 0)} + 1_{(X_2+X_3\neq 0,\, X_3\neq 0)} + ... + 1_{(X_{n-1}+X_n\neq 0,\, X_n\neq 0)}\}$$

In the *R.H.S.* of above inequality, if we let

$$\begin{array}{rcl} A_1 &=& 1_{(X_1+X_2\neq 0,\, X_2\neq 0)} \\ A_2 &=& 1_{(X_2+X_3\neq 0,\, X_3\neq 0)} \\ .. && \\ A_{n-1} &=& 1_{(X_{n-1}+X_n\neq 0,\, X_n\neq 0)} \end{array}$$

then we can see sequence $(A_1, A_2, A_3, ...)$ has the form of $A_i = \varphi(X_i, X_{i+1}, X_{i+2})$. Observe that the shift $(X_1, X_2, X_3, ...) \to (X_2, X_3, X_4, ...)$ is by assumption measure-preserving and ergodic, hence Ergodic Theorem applies, i.e.

$$\limsup_{n\to\infty} R_n/n \le P(X_1 + X_2 \neq 0,\, X_2 \neq 0)\, a.s. \tag{1}$$

Lemma. If $X_1, X_2, ...$ is a stationary ergodic sequence and φ is a product-measurable function, and let $Y_n = \varphi(X_n, X_{n+1}, X_{n+2}, ...)$, then $(Y_1, Y_2, Y_3, ...)$ is stationary and ergodic.

Now observe that

$$P(X_1 + X_2 \neq 0,\, X_2 \neq 0) = P(X_1 \neq 0,\, X_1 + X_2 \neq 0)$$

because $(X_1, X_2) =_d (X_2, X_1)$ in *I.I.D.* case. Continuing like this, we get

$$\limsup_{n\to\infty} \frac{R_n}{n} \le P(X_1 + ... + X_k \neq 0,\, X_2 + ... + X_k \neq 0, ...,\, X_{k-1} + X_k \neq 0,\, X_k \neq 0)$$

Staying with the case X_i's are *I.I.D.*, i.e., $(X_n, X_{n-1}, ... X_1) =_d (X_1, X_2, ..., X_n)$, we can see that the *R.H.S.* of above inequality is equivalent to $P(S_1 \neq 0, S_2 \neq 0, ..., S_k \neq 0)$. Now let $k \to \infty$, we get

$$\limsup_{n\to\infty} \frac{R_n}{n} \le P(S_1 \neq 0,\, S_2 \neq 0,\, S_3 \neq 0, ...)\, a.s.$$

Look at the lower bound and compare first and last term, we have

$$R_n = 1_{(S_{n-1}\neq S_n)} + 1_{(S_{n-2}\neq S_{n-1},\, S_{n-2}\neq S_n)} + ... + 1_{(0\neq S_1,\, 0\neq S_2, ...,\, 0\neq S_n)}$$

This implies the following:

$$\begin{array}{rcl} \frac{R_n}{n} &\ge& (1/n)\sum_{k=0}^{n} 1_{(S_k\neq S_{k+1},\, S_k\neq S_{k+2}, ...,\, S_k\neq S_{n+k})} \\ &=& (1/n)\sum_{k=0}^{n} 1_{(X_{k+1}\neq 0,\, X_{k+1}+X_{k+2}\neq 0, ...)} \\ &\to& P(X_1 \neq 0,\, X_1 + X_2 \neq 0, ...)\, a.s. \end{array}$$

by Ergodic Theorem.

Hence we have

$$\liminf_{n\to\infty} \frac{R_n}{n} = P(X_1 \neq 0,\ X_1 + X_2 \neq 0, ...)$$

Combining this with (1) yields:

$$\begin{aligned} \lim_{n\to\infty} \tfrac{R_n}{n} &= P(X_1 \neq 0,\ X_1 + X_2 \neq 0, ...) \\ &= P(S_1 \neq 0,\ S_2 \neq 0,\ S_3 \neq 0, ...) \end{aligned}$$

Therefore we have proved the K-S-W Theorem. □

Alternate approach to prove Ergodic Theorem

Given $(\Omega,\ \mathcal{F},\ P,\ T)$ with T preserving P, assume $E|X| < \infty$. The proof of the ergodic theorem used the following *key inequality*: for $a \geq 0$

$$E(X \cdot 1(\limsup_{n\to\infty} S_n/n \geq a)) \ \geq\ aP(\lim_{n\to\infty} \sup \frac{S_n}{n} \geq a)$$

which obviously implies the following:

$$\text{If } P(\limsup_{n\to\infty} S_n/n \geq a) = 1 \text{ then } EX \geq a \tag{2}$$

Notice that (2) also implies the key inequality. Let

$$C := (\limsup_{n\to\infty} S_n/n \geq a)$$

The only interesting case is when $P(C) > 0$. But then we can condition on C to make a new setup $(\Omega^C,\ \mathcal{F}^C,\ P^C,\ T^C)$ with:

$$\begin{aligned} \Omega^C &:= C \in F \\ \mathcal{F}^C &:= \{A \subset C :\ A \in \mathcal{F}\} \\ P^C(\cdot) &:= P(\cdot|C) := P(. \cap C)/P(C) \\ T^C(\omega) &:= T(\omega) \text{ for } \omega \in C. \end{aligned}$$

We can easily check that this transformation T^C preserves P^C. The key inequality is recovered by applying (2) to $(\Omega^C, \mathcal{F}^C, P^C, T^C, X^C)$ with $X^C = X$ restricted to C.

Alternative proof for the key inequality due to Paul Shields [1].

Proof. We break the proof into two steps.
Step 1: Fix $a \geq 0$. Let $0 \ll N$. Looking along an orbit of $S_n(\omega)/n$:

We put a mark, i.e., "x", if $T^n(\omega) \in B_L$, where $B_L = \{S_n/n \leq a :\ 1 \leq n \leq L\}$.

Loops in the figure are defined to be successive intervals of length at most L over which the average of X_i's is greater than a. For insance, in the figure, $\frac{X_{10}+X_{11}+X_{12}}{3}(\omega) > a$.

Thus

$$\begin{aligned} \tfrac{X_0+X_1+...+X_{N-1}}{N} &\geq \tfrac{a}{N}(\text{ Length of stretches where average } > a) \\ &\quad + \tfrac{1}{N}(\text{ sum of the } X_i \text{ over the rest of the orbit }) \\ &\geq \tfrac{a}{N}(N - \textstyle\sum_{n=0}^{N-1} 1_{(T^n \in B_L)} - L) \\ &\quad + \tfrac{1}{N} \textstyle\sum_{n=0}^{N-L-2} X_n \cdot 1_{(T^n \in B_L)} - \tfrac{1}{N} \textstyle\sum_{n=N-L-1}^{N-1} |X_n| \end{aligned}$$

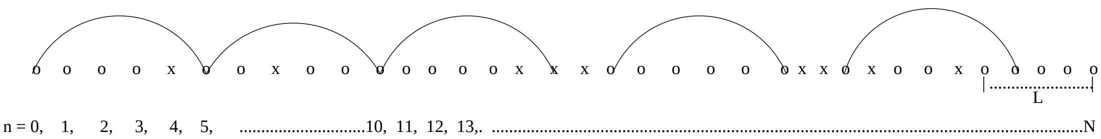

Figure 1: Orbit of S_n/n.

Because T is a measure-preserving transformation, we obtain the following from the inequality above by taking expected value at both sides:

$$EX \geq a - aP(B_L) - \frac{aL}{N} + \frac{N-L-1}{N} E(X \cdot 1_{B_L}) - \frac{L}{N} E|X|$$

Since this is true for all N, we can let $N \to \infty$ to deduce

$$EX \geq a - P(B_L)a + E(X \cdot 1_{B_L}).$$

But if $P(\limsup_{n\to\infty} S_n/n > a) = 1$, then $B_L \downarrow \emptyset$ a.s., and it follows by dominated convergence that $EX \geq a$, that is (2). $\square$

References

[1] P. C. Shields. The ergodic and entropy theorems revisited. *IEEE Trans. Inform. Theory*, 33(2):263–266, 1987.