Stat205B: Probability Theory (Spring 2003)	**Lecture: 10**
The Ergodic Theorem	
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11 Preliminaries and notations

The Ergodic theorem presented here is due to Birkoff (1931). The proof was simplified by various people, including Garcia (1965).

T:$\Omega \to \Omega$ is a measure preserving transformation on $(\Omega, \mathcal{F}$,P). Also P(w:T(w) $\in$ B) $\forall$ B $\in \mathcal{F}$. Also let X $\in L^1$.

12 Ergodic theorem

$$\frac{1}{n}\sum_{k=0}^{n-1} X \circ T^k \longrightarrow E(X/\mathcal{I}) \quad a.s \quad in\ L^1 \tag{1}$$

where $\mathcal{I}$ is the invariant σ - field. i.e, $\mathcal{I} = \{B \in \mathcal{F} : B = T^{-1}(B)\}$

13 Proof

First check that a random variable Y is $\mathcal{I}$ - measurable iff Y is $\mathcal{F}$ measurable and Y $\circ$ T = Y. Let $S_n = \sum_{k=0}^{n-1} X \circ T^k$ and $S_0 = 0$. Also let $M_n = \max\{0, S_1, S_2, S_3, ..., S_n\}$. Note that $M_n \geq 0$ The major steps in the proof are outlined in the following steps:

- *Step 1*: Prove $E(X\ 1(M_n \geq 0)) \geq 0$
- *Step 2*: Prove For a>0, $E(X\ 1(\overline{\lim}\ \frac{S_n}{n} > a)) \geq aP(\overline{\lim}\ \frac{S_n}{n} > a)$
- *Step 3*: Reduction of the general problem to a case where $E(X/\mathcal{I}) = 0$
- *Step 4*: If $E(X/\mathcal{I}) = 0$, then $\frac{S_n}{n} \to 0\ a.s$

Step 1: Prove $\mathbf{E(X\ 1(M_n \geq 0)) \geq 0}$

Proof:

If $M_n > 0$, then $M_n = X + M_{n-1} \circ T$

If $M_n = 0$, then obviously $M_{n-1}(T) \geq 0$ and $M_n \leq M_{n-1} \circ T$

From the above two statements, for all ω, the following is true:

$$M_n \leq X1(M_n > 0) + M_{n-1} \circ T \tag{2}$$

It is quite obvious that $M_{n-1} \leq M_n$, therefore the above equation leads to the following one:

$$M_n \leq X1(M_n > 0) + M_n \circ T \tag{3}$$

hence,

$$E(M_n) \leq E(X1(M_n > 0)) + E(M_n \circ T) \tag{4}$$

but $E(M_n \circ T) = E(M_n)$ because T preserves P. This leads to the result of this step, that is

$$E(X1(M_n \geq 0)) \geq 0 \tag{5}$$

Step 2: Prove For a>0, E(X 1($\overline{\lim}\ \frac{S_n}{n} > a$)) $\geq$ aP($\overline{\lim}\ \frac{S_n}{n} > a$)

Proving the result in step 2 is the same as proving the following result:

For a>0, E((X-a) 1($\overline{\lim}\frac{S_n}{n} > a$)) ≥ 0)

Let $X^* = X-a$; then $S_n^* = \sum_{k=0}^{n-1} X^* \circ T^k = S_n - na$.

Hence it suffices to prove the following result for this step:

$$For \quad a > 0, \quad E(X^*\ 1(\overline{\lim}\ \frac{S_n^*}{n} > 0))\ \geq\ 0) \tag{6}$$

Let n tend to ∞ in the result of step 1, then we get,

$$E(X\ 1(S_n > 0\ for\ some\ n))\ \geq\ 0 \tag{7}$$

Now the idea to complete the proof of step 2 is to replace X in the above equation by $\mathbf{X} = X\ 1(\ \overline{\lim}\ \frac{S_n}{n} > 0)$. Also check that $\overline{\lim}\ \frac{S_n}{n}$ is a invariant random variable. So, after substituting we have

$$E(\mathbf{X}\ 1(\mathbf{S}_n > 0\ for\ some\ n))\ \geq\ 0 \tag{8}$$

Substituting for $\mathbf{X}$, we have

$$E(X\ 1(\overline{\lim}\ \frac{S_n}{n} > 0)\ 1(S_n > 0\ for\ some\ n)\ 1(\overline{\lim}\ \frac{S_n}{n} > 0))\ \geq\ 0. \tag{9}$$

This gives what we wanted to prove in equation 6 as the above equation results in the following one:

$$E(X\ 1(\overline{\lim}\ \frac{S_n}{n} > 0))\ \geq\ 0) \tag{10}$$

Step 3: Reduction of the general problem to case where $\mathbf{E(X/\mathcal{I}) = 0}$

This is done by replacing X by $\hat{X} := X - E(X/\mathcal{I})$ in the ergodic theorem. Then $\hat{S}_n = S_n - nE(X/\mathcal{I})$. So $\frac{\hat{S}_n}{n} \to 0$ a.s iff $\frac{S_n}{n} \to$E(X/$\mathcal{I}$) a.s. Hence it is enough to treat the case when $E(X/\mathcal{I}) = 0$..

Step 4: Final nail in the coffin: If $\mathbf{E(X/\mathcal{I}) = 0}$, then $\frac{S_n}{n} \to 0 \ a.s$

Consider the left hand side of the result from step 2. Now E(X 1($\overline{\lim}\ \frac{S_n}{n} >$ a)) = E[E(X 1($\overline{\lim}\ \frac{S_n}{n} >$ a))/$\mathcal{I}$] = E[E(X/$\mathcal{I}$) 1($\overline{\lim}\frac{S_n}{n} >$ a))] = 0 because E(X/$\mathcal{I}$) = 0.

So, if E(X/$\mathcal{I}$) = 0, then for all a > 0 we have the following result:

$$0 \geq aP(\overline{\lim}\ \frac{S_n}{n} > a) \tag{11}$$

or

$$aP(\overline{\lim}\ \frac{S_n}{n} > a) = 0 \tag{12}$$

Therefore, $\overline{\lim}\ \frac{S_n}{n} \leq 0$ a.s. Also replacing X by -X, and by the same line of reasoning we have, $\underline{\lim}\ \frac{S_n}{n} \geq 0$ a.s. Hence,

$$\frac{S_n}{n} \to 0 \ as \ n \to \infty \ a.s \tag{13}$$