Stat205B: Probability Theory (Spring 2003)	**Lecture: 1**

Introduction to Markov Chains

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We begin this course with the theory of Markov chains. Let $(S, \mathcal{S})$, be any measurable space. Usually S is a finite set, a countable set, or $\mathbb{R}^n$. For the most part we will confine our attention to discrete time processes. In the continuous time, continuous state space setting, Markov chains are known as Markov Processes.

1.1 Markov Property and Existence

Definition 1.1 *A sequence of random variables (X_n) is called a* **Markov chain** *if the past and future of the process are conditionally independent given the present.*

Example 1.1 *A random walk is an example of a Markov chain.*

Definition 1.2 *A function $p : S \times \mathcal{S} \to \mathbb{R}$ is called a* **Markov kernel** *if*

1) For each $x \in S$, the mapping $A \to p(x, A)$ is a probability distribution on $(S, \mathcal{S})$.

2) For each $A \in \mathcal{S}$, the mapping $x \to p(x, A)$ is an $\mathcal{S}$-measurable function.

Definition 1.3 *The Markov kernel p_n is called a* **transition probability function***, or t.p.f, for (X_n) if*

$$P(X_{n+1} \in B|\mathcal{F}_n) = p_n(X_n, B)$$

for each $B \in \mathcal{S}$.

In other words, $p_n(x, B)$ is the probability that the next step in the chain lies in B given that the current state is x. In the absence of the subscript n we call p a homogeneous transition probability function. Henceforth, in these notes we assume that (X_n) has the homogeneous t.p.f. p.

Theorem 1.1 *(Ionescu-Tulcea) Given a measuable space $(S, \mathcal{S})$ with distribution μ, and a transition probability function p, there exists a Markov chain on the space and its distribution, P_μ, is unique on $(S \times S \times \ldots, \mathcal{S}^\infty)$. Here $\mathcal{S}^\infty$ is the product σ-field.*

It often convenient to suppose that (X_n) is a coordinate process. That is, for $w \in \Omega = \{(w_0, w_1, \ldots) : w_i \in S\}$ we set

$$X_n(w) = w_n$$

Proof Sketch: Under regularity assumptions on S this is a consequence of Kolmogorov's Extension Theorem.

Notice that if we define,

$$\begin{aligned}
P_\mu(X_0 \in A_0) &= \mu(A_0) \\
P_\mu(X_0 \in A_0, X_1 \in A_1) &= \int_{A_0} \mu(dx_0)p(x_0, A_1) \\
P_\mu(X_0 \in A_0, X_1 \in A_1, X_2 \in A_2) &= \int_{A_0} \mu(dx_0) \int_{A_1} p(x_0, dx_1)p(x_1, A_2)
\end{aligned}$$

and so on, then we have a sequence of distributions on $S, S \times S, \dots$ that is consistent in the sense of Kolmogorov's Extension Theorem. Measure theory then tells us that there exists a distribution on $S \times S \times \dots$ such that the first n coordinates are distributed as above on S^n. ■

1.2 Some General Facts

To find the distribution of X_n we first regard the Markov kernel, $p(\cdot, \cdot)$ as an operator on measures,

$$\mu p(B) := \int \mu(dx)p(x, B)$$

Thus μp is a new probability distribution. It is the distribution of X_1 for a Markov chain, $(X_0, X_1, \dots)$, with $X_0 \sim \mu$ and t.p.f, p. Similarly,

$$\mu p^n(B) = \int \mu(dx)p^n(x, B) = \text{distribution of } X_n$$

where $p^n(x, B) = P_\mu(X_n \in B | X_0 = x)$.

When S is countable we typically denote the elements of S by i, j, k, etc. In this case we define the transition matrix, P, by

$$P_{ij} = p(i, \{j\})$$

the probability of transitioning from state i to state j given that the current position is i. We can also identify the initial distribution, μ with a row vector,

$$\mu_i = \mu(\{i\})$$

Clearly the matrix P must satisfy, $\sum_j P_{ij} = 1$, for each $i \in S$.

Applying this notation to the discussion at the begining of the section we conclude that if (X_n) is a Markov chain with countable state space transition matrix P then

$$P_\mu(X_n = j | X_0 = i) = P^n_{ij}$$

and if $X_0 \sim \mu$, and P^n denotes the nth matrix power of P, then

$$\mu P^n = \text{distribution of } X_n.$$

On a general state space $(S, \mathcal{S})$ and for a suitable $f : S \to \mathbb{R}$, say bounded measurable or non-negative measurable, define

$$pf(x) := \int_S f(y)p(x, dy)$$

Claim 1.2

$$\begin{aligned} pf(x_n) &= E_\mu[f(X_{n+1})|X_n = x_n] \\ &= E_\mu[f(X_{n+1})|X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0] \end{aligned}$$

Proof: See Durrett, section 5.1. ■

Similarly we have that

$$p^m f(x_n) = E_\mu[f(X_{n+m})|X_n = x_n]$$

In the case that S is countable the action of p on f can again be interpreted as a matrix vector operation, Pf.

Now consider those functions h such that $ph = h$. These functions are called *harmonic functions* because of a close relationship with the harmonic functions of Analysis. Appling the result of the claim, if h is harmonic then

$$E_\mu[h(X_{n+1})|X_n = x_n] = ph(x_n) = h(x_n)$$

Thus for any intial distribution μ, $(h(X_n))$ is an $\mathcal{F}_n$-martingale.

Example 1.2 Let B be the set of all *absorbing states*, meaning that $p(b, b) = 1$ for all $b \in B$. Call B the *boundary* of the state space S. Let A be some subset of B and define

$$h_A(x) = P_\mu(X_n \in A \text{ eventually}).$$

Then (see Durrett, Section 5.2, Exercise 2.6)

1) h_A is a p-harmonic function.
2) if $P_x(X_n \in B \text{ eventually}) = 1$ then h_A is the unique p-harmonic function whose boundary values are given by 1_A, the indicator function of A.