Stat205B: Probability Theory (Spring 2003) **Lecture: 22**

Brownian Bridge

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Setup: $X_1, X_2, \cdots$ i.i.d. according to some F="theoretical distribution"

Definition 22.1 $F_n(x,\omega) := \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i(\omega) \le x)$

Assume for simplicity that F is continuous.
Look at the Kolmogorv-Smirnov statistic:

$$D_n := \sqrt{n}\sup_x |F_n(x) - F(x)|$$

Then by CLT for Binomial case

$$\sqrt{n}\sup_x |F_n(x) - F(x)| \xrightarrow{\text{d}} N(0, F(x)[1-F(x)]), \quad \text{for each x.}$$

What about as x varies ?
Take x, y : we get two correlated Gaussian by application of multidimensional CLT.
So we expect there is a limiting Gaussian process, s.t.

$$\sqrt{n}\sup_x |F_n(x) - F(x)| \xrightarrow{\text{d}} G(x)$$

$$D_n \xrightarrow{\text{d}} \sup_x G_n(x)$$

A simplification:
We really don't need to deal with $F(x)$. Transform the data by

$$X_i \longrightarrow F(X_i) \stackrel{\text{def}}{=} U_i$$

Then

1) $U_1, U_2, \cdots$ are i.i.d. and in $U[0,\ 1]$
2) $D_n(X_1, X_2, \cdots, X_n, F) = D_n(U_1, U_2, \cdots, U_n, U[0,\ 1])$

So we may as well work with $U_1, U_2, \cdots$ i.i.d. and in $U[0,\ 1]$, i.e.

$$D_n = \sup_t |H_n(t)|$$

where

$$H_n(t) := \sqrt{n}\,(\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(U_i \le t) - t)$$

And

$$EH_n(t) = 0, \quad VarH_n(t) = t(1-t)$$

$$\begin{aligned} E(H_n(s)H_n(t)) &= Cov(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}(U_i \le s), \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\mathbf{1}(U_j \le t)) \\ &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} Cov(\frac{1}{\sqrt{n}}\mathbf{1}(U_i \le s), \frac{1}{\sqrt{n}}\mathbf{1}(U_j \le t)) \\ &= Cov(\frac{1}{\sqrt{n}}\mathbf{1}(U_i \le s), \frac{1}{\sqrt{n}}\mathbf{1}(U_j \le t)) \\ &= s(1-t) \end{aligned}$$

Summary: The FDD's of $(H_n(t), 0 \le t \le 1) \xrightarrow{\text{d}}$ FDD's of $(B^0(t), 0 \le t \le 1)$ where $B^0(t)$ is a Gaussian process with $EB^0(t) = 0, \;\; E(B^0(s)B^0(t)) = s(1-t)$

Theorem 22.1 *There exits a version of this Gaussian process with continuous path. Moreover, such a process can be constructed in various ways for Brownian motion B:*

$$B^0(t) = B(t) - tB(1)^1, \quad 0 \le t \le 1$$

and $(B^0(t), 0 \le t \le 1)$ and B(1) are independent.

Proof Sketch: It's easy to check that

$$EB^0(t) = 0$$

$$EB^0(s)B^0(t) = s(1-t) \text{ for } 0 < s < t$$

using $EB(s) = 0, \; EB(s)B(t) = s \wedge t$.

$$\begin{aligned} E(B^0(s)B(1)) = 0 \quad &\Rightarrow \quad E((\sum_i a_i B^0(s_i))B(1)) = 0 \\ &\Rightarrow \quad B(1) \text{is independent of } B^0(s), \;\; s \in \text{finite set} \\ &\Rightarrow \quad B(1) \text{ is independent of } B^0(s) \text{ for } 0 \le s \le 1 \end{aligned}$$

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Theorem 22.2 $D_n = \sup_{0 \le t \le 1} |H_n(t)| \xrightarrow{\text{d}} \sup_{0 \le t \le 1} |B^0(t)|$

Remark: B^0 should be understood informally as $B|B(1) = 0$.
Rigorously, $(B(t), 0 \le t \le 1||B(1)| < \epsilon) \xrightarrow{\text{d}} (B^0(t), 0 \le t \le 1)$ in the first instance. This is for FDD's. It's also true in C[0, 1].
Remark2: It's also true that

$$(H_n(t), 0 \le t \le 1) \xrightarrow{\text{d}} (B^0(t), 0 \le t \le 1)$$

in sense of FDD's. But it's false in C[0,1] for a trivial reason($H_n(.)$ has jumps!)
What can you do?

[1] So $E(B^0(s))^2 = s(1-s) \approx s \; as \; s \approx 0$

Definition 22.2 $D[0,1] :=$ *space of path which is right-continuous with left limits.*

Put a suitable topology . Then get $\xrightarrow{\mathrm{d}}$ for process with paths in D[0,1].

Proof Sketch:[2]
$\sup_{0\leq t\leq 1} H_n(t)$ is a function of the order statistic $U_{n,1}, U_{n,2}, \cdots, U_{n,n}$ of $U_1, U_2, \cdots, U_n$

$$\sup_{0\leq t\leq 1} H_n(t) = \text{ max of n values computed from} U_{n,1}, U_{n,2}, \cdots, U_{n,n}$$

General tool for handling formula for functions of $U_{n,1}, U_{n,2}, \cdots, U_{n,n}$:
Let $w_1, \cdots, w_n$ be i.i.d. with $P(w_i > t) = e^{-t}$ and let $z_n = w_1 + \cdots + w_n$. Then

$$(U_{n,1}, U_{n,2}, \cdots, U_{n,n}) \stackrel{\mathrm{d}}{=} (\frac{z_1}{z_{n+1}}, \cdots, \frac{z_n}{z_{n+1}})$$

Now $\sup_{0\leq t\leq 1} H_n(t) \stackrel{\mathrm{d}}{=}$some function of $z_1, \cdots, z_{n+1}$. Apply Dunker's theorem to $z_1, z_2, \cdots$ to deduce

$$\sup_{0\leq t\leq 1} H_n(t) \stackrel{\mathrm{d}}{=} \sup_{0\leq t\leq 1} B^0(t)$$

and $B^0(t) = B(t) - tB(1)$. ∎

Alternate construction of $B^0(t)$:
Let $\widehat{B}(t) = h(t)B(\frac{t}{1-t})$.
Idea: make paths of $\widehat{B}(t) \stackrel{\mathrm{d}}{=} B^0(t)$ by using a space time change

$$[0,1] \longleftrightarrow [0,\infty)$$

$$t \longrightarrow \frac{t}{1-t}$$

Now choose h(t) s.t. $\widehat{B}(t) \stackrel{\mathrm{d}}{=} B^0(t)$.
Check the variance:

$$h^2(t)\frac{t}{1-t} = t(1-t) \Rightarrow h(t) = 1-t$$

Prove that it's a bridge:
Notice that $0 \leq s \leq t < 1 \Rightarrow \frac{s}{1-s} \leq \frac{t}{1-t}$. So

$$\begin{aligned} E(\widehat{B}(s)\widehat{B}(t)) &= E[(1-s)(1-t)B(\frac{s}{1-s})B(\frac{t}{1-t})] \\ &= (1-s)(1-t)\frac{s}{1-s} \\ &= s(1-t) \end{aligned}$$

Remark: This a a good transformation because it pushes lines to lines, crossings to crossings.

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[2]see textbook for details