Lecture 26 : Poisson Point Processes

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In this lecture, we consider a measure space $(S, \mathcal{S}, \mu)$, where μ is a σ-finite measure (i.e. S may be written as a disjoint union of sets of finite μ-measure).

26.1 The Poisson Point Process

Definition 26.1 *A* **Poisson Point Process** *(P.P.P.) with intensity μ is a collection of random variables $N(A, \omega)$, $A \in \mathcal{S}$, $\omega \in \Omega$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that:*

1. *$N(\cdot, \omega)$ is a counting measure on $(S, \mathcal{S})$ for each $\omega \in \Omega$.*
2. *$N(A, \cdot)$ is Poisson with mean $\mu(A)$:*

$$\mathbb{P}(N(A) = k) = \frac{e^{-\mu(A)}(\mu(A))^k}{k!} \text{ all } A \in \mathcal{S}.$$

3. *If A_1, A_2, ... are disjoint sets then $N(A_1, \cdot)$, $N(A_2, \cdot)$, ... are independent random variables.*

Theorem 26.2 *P.P.P.'s exist.*

Proof Sketch: It's not enough to quote Kolmogorov's Extension Theorem. The only convincing argument is to give an explicit constuction from sequences of independent random variables. We begin by considering the case $\mu(S) < \infty$.

1. Take X_1, X_2, ... to be i.i.d. random variables so that $\mathbb{P}(X_i \in A) = \frac{\mu(A)}{\mu(S)}$.
2. Take $N(S)$ to be a Poisson random variable with mean $\mu(S)$, independent of the X_i's. Assume all random varibles are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
3. Define $N(A) = \sum_{i=1}^{N(S)} \mathbf{1}_{(X_i \in A)}$, for all $A \in \mathcal{S}$.

Now verify that this $N(A, \omega)$ is a P.P.P. with intensity μ. If $\mu(S) = \infty$ we write $S = S_1 \bigcup_{i=1}^{\infty} S_i$ as a disjoint union where $\mu(S_i) < \infty$ and construct P.P.P.'s $N_i(\cdot)$ with intensity μ restricted to S_i. Make the $N_i(\cdot)$ independent and define $N(A) = \sum_{i=1}^{\infty} N_i(A)$ for all $A \in \mathcal{S}$. The superposition and thinning properties of Poisson random variables now imply that $N(\cdot)$ has the desired properties[1]. ■

The most common way to construct a P.P.P. is to define

$$N(A) = \sum_i \mathbf{1}_{(T_i \in A)} \tag{26.1}$$

for some sequence of random variables T_i which are called the points of the process.

[1] For a reference, see *Poisson Processes*, Sir J.F.C. Kingman, Oxford University Press.

Example 26.3 *Let μ be Lebesgue measure on $[0, \infty)$. Define random variables T_i so that $0 < T_1 < T_2 < T_3 < ...$ with $T_1 = W_1$, and $T_r = W_1 + ... + W_r$ where the W_i are i.i.d. exponential(λ) random variables. That is, $\mathbb{P}(W_i > t) = e^{-\lambda t}$ and if we define $N_t = N([0, t))$ we find that*

$$\mathbb{P}(N_t = j) = \frac{e^{-\lambda t}(\lambda t)^j}{j!} \tag{26.2}$$

26.2 An Application

We now describe a general method for constructing a process with independent increments from a P.P.P. In particular, we wish to construct a process $(X_t; t \geq 0)$ of the form

$$X_t = \sum_{0<s\leq t} \Delta X_s \tag{26.3}$$

where the above sum has only countably many non-zero terms, and the collection $\{(s, \Delta X_s) : s > 0, \Delta X_s \neq 0\}$ is the set of points of a P.P.P. on $(0, \infty) \times \mathbb{R} - \{0\}$ with intensity measure $m \times L$ for some Levy-measure L on $\mathbb{R} - \{0\}$ (m denotes Lebesgue measure).

Example 26.4 *A generalized Poisson Process may be constructed as follows. Let N_1, N_2, ..., N_m be independent P.P.P.'s with rate λ_i. Define*

$$X_t = \sum_{i=1}^{m} a_i N_i(t) \tag{26.4}$$

so that X_t jumps by a_i whenever N_i jumps by 1. In this case $\Delta X_t = \sum_i a_i \Delta N_i(t)$ and the Levy-measure is

$$L(A) = \sum_{i=1}^{m} \lambda_i 1_{(a_i \in A)}. \tag{26.5}$$

The familiar Poisson Process with parameter λ is obtained by letting $m = 1$, $\lambda_1 = \lambda$ and $a_1 = 1$.

This example illustrates the concept for a discrete Levy-measure L. From the previous lecture, we can handle a general finite measure L by setting

$$X_t = \sum_{i=1}^{\infty} Y_i 1_{(T_i \leq t)} \tag{26.6}$$

where the T_i are the points of jumps of a standard Poisson Process with rate $L(\mathbb{R})$ and the Y_i are i.i.d. with $\mathbb{P}(Y_i \in A) = \frac{L(A)}{L(\mathbb{R})}$. If L is supported on $(0, \infty)$, i.e. we only allow positive jumps, we can compute:

$$\begin{aligned}
E\left[e^{-\theta X_t}\right] &= \sum_{i=1}^{\infty} \mathbb{P}(N_t = i)(E\left[e^{-\theta Y}\right])^i && (26.7)\\
&= \sum_{i=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^i}{i!}\left(E\left[e^{-\theta Y}\right]\right)^i && (26.8)\\
&= e^{-\lambda t} exp\left[\lambda t E\left[e^{-\theta Y}\right]\right] && (26.9)\\
&= e^{-\lambda t} exp\left[t\int_0^{\infty} e^{-\theta x} L(dx)\right] && (26.10)\\
E\left[e^{-\theta X_t}\right] &= exp\left[t\int_0^{\infty}\left(e^{-\theta x} - 1\right) L(dx)\right]. && (26.11)
\end{aligned}$$

To summarize: we have shown that if $(X_t, t \geq 0)$ is a process with independent increments where the set of jumps $((s, \Delta X_s), s > 0)$ is the set of points of a P.P.P. on $(0, \infty) \times \mathbb{R}$ with intensity distribution $L(dx)$, then X_t has distribution specified by the Laplace Transform above (26.11). This formula is an instance of the Levy-Khintchine equation.

Example 26.5 *Consider two independent processes X_t and Y_t which correspond to P.P.P.'s with measures $dt \times L(dx)$ and $dt \times M(dx)$ respectively. Then $X_t + Y_t$ is a P.P.P. with measure $dt \times (M + L)(dx)$ as one may check via direct computation using equation (26.11).*

These results may also be extended to suitable infinite measures. Suppose we can write $L = \sum_i L_i$ where the L_i's are finite and $\int_0^\infty xL(dx) < \infty$. Then we may define P.P.P.'s X_i according to the measures $dt \times L_i(dx)$ and sum them to obtain a process X with jumps according to the measure $dt \times L(dx)$.

Finally, it is of interest to ask for which measures L equation (26.11) give the Laplace Transform of a distribution on $[0, \infty]$? One may show that the following three conditions are equivalent:

1. Equation (26.11) is the Laplace Transform of a distribution on $[0, \infty]$
2. $\int_0^\infty \left(1 - e^{-\theta x}\right) L(dx) < \infty$ for all $\theta < \infty$
3. $\int_0^\infty \left(1 - e^{-\theta x}\right) L(dx) < \infty$ for some $\theta < \infty$.

Exercise 26.6 *Check that if you make this Poisson construction of jumps with $L(dx) = x^{-1}e^{-x}dx$ then X_t = gamma(t). That is:*

$$\mathbb{P}(X_t \in dx) = \frac{1}{\Gamma(t)} e^{-x} x^{t-1} dx. \tag{26.12}$$