Lecture 18 : Stopping Times and Martingales

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18.1 Stopping Times

Assume that $\{\mathcal{F}_n\}$ is an increasing set of $\sigma-$fields. Recall that a stopping time T is a random variable $T : \Omega \leftarrow \mathbb{Z}^+$ such that $\forall\ n < \infty$, one of the following equivalent conditions holds -

1. $\{T = n\} \in \mathcal{F}_n$
2. $\{T \leq n\} \in \mathcal{F}_n$
3. $\{T \geq n\} \in \mathcal{F}_n$

Walds Identity - Let $X_1, X_2, X_3, ...$ be i.i.d. random variables with $E|X_i| < \infty$ and T be a stopping time of $\{\mathcal{F}_n\}$ where $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$ with $E(T) < \infty$. Let $S_n = X_1 + X_2 + ... + X_n$. Then $ES_T = EX_1ET$.

Proof:

$$ES_T = E(\sum_1^T X_i) = \sum_1^\infty E(X_i 1_{T \geq i}) = \sum_1^\infty E(X_i 1_{T > i-1}) = \sum_1^\infty EX_i E1_{T > i-1} = EX_1 ET$$

The final step is because X_i and $1_{T>i-1}$ are independent $\square$.

While it seems as if common means for the $X_i s$ suffices instead of identical distribitions. However this is not true. The reason for this is that we need $E|X_i| < \infty$ to hold without which the summation and integral (in the calculation of expected value of S_T) cannot be exchanged. The following example illustrates this.

Define X_i as $P(X_i = \pm 2^i) = \frac{1}{2}$. Let $T = \{\inf n : S_n \geq 1\}$. Clearly, $P(T = n) = \frac{1}{2}^n$, $ET = 2 < \infty$ and $ES_T \geq 1$. However, $EX_i = 0$ which clearly violates Walds identity.

We now derive the classic **Gamblers Ruin** formula using Walds identity. The problem is that of a random walk that starts at $X_0 = a > 0$ and X_i is a symmetric simple random walk i.e., a probability of $\frac{1}{2}$ for both 1 and -1. Define $T = \{\inf n : a + S_n = 0 \ or \ a + S_n = b\}$. A practical explanation of this problem is that of a gambler starting with a capital of a. We are interested in the probability that the gambler wins b before going broke. Formally, we want to calculate $P(a + S_T = b)$. Now, $E(a + S_T) = bP(a + S_T = b)$ is also given by $a + ES_T$ which is a by Walds identity. Hence, the gambler earns b before getting broke with a probability of $\frac{a}{b}$.

18.2 Martingales

Martingales are defined for a filtration i.e., an increasing sequence of $\sigma-$fields, $\mathcal{F}_n$ $(n = 1, 2....)$. A sequence of random variables M_n is adapted to this filtration of $M_n \in \mathcal{F}_n$ $(n = 1, 2 \ldots)$. Such a filtration is a martingale (MG) (w.r.t. $\mathcal{F}_n$) if

- $E|M_n| < \infty$ and
- $E(M_{n+1}|\mathcal{F}_n) = M_n \forall n$.

We define $M_0 = 0$ for convenience and use this definition unless explicitly mentioning otherwise in the rest of the course. The filtration with finite means is a sub-martingale if $E(M_{n+1}|\mathcal{F}_n) \geq M_n$ and a super-martingale if $E(M_{n+1}|\mathcal{F}_n) \leq M_n \forall n$. Note that in the case of martingales, the second condition implies that M_n is a filtration whereas this is not true in the case of sub-martingales and super-martingales. Another definition that will be used later is that of a predictable sequence. A predictable sequence of random variables M_n such that $M_n \in \mathcal{F}_{n-1}$.

An example of a MG is $S_n = \sum_{i=1}^{n} X_i - nEX_1$ where X_i is a sequence of i.i.d random variables. This is because

$$E(S_{n+1}-(n+1)EX_1|\mathcal{F}_n) = E(S_n-(n+1)EX_1|\mathcal{F}_n)+E(X_{n+1}|\mathcal{F}_n) = S_n-(n+1)EX_1+EX_{n+1} = M_n$$

Also notice that if $X_n = M_n - M_{n-1}$ then $E(X_n|\mathcal{F}_{n-1}) = 0$. Similar results can easily be derived for super-martingales and sub-martingales. We will be considering two kinds of results involoving MGs. These are optional stopping theorems (maximal inequalities) and convergence theorems.

Martingales and predictable sequences can be used in a natural way in gambling systems. If X_n is the outcome of the n^{th} bet and H_n is the multiplier that the gambler places for this bet his/her earnings on this bet are $X_n \cdot H_n$. Since gamblers can place bets at time n based upon the outcomes at times $1 ... n-1$, $H_n \in \mathcal{F}_{n-1}$ i.e., H_n is predictable and S_n is a martingale. Denoting the gamblers earnings after n bets as a new variable Y_n we get, assuming $X_0 = 0$,

$$Y_n = H_1 \cdot X_1 + \ldots H_n \cdot X_n = H_1(S_1 - S_0) + \cdots + H_n(S_n - S_{n-1}) = (H \cdot S)_n \tag{18.1}$$

which is the MG-transform. Y_n is an $\mathcal{F}_n$-martingale if $E(Y_n - Y_{n-1}|\mathcal{F}_n) = 0$. But,

$$E(Y_n - Y_{n-1}|\mathcal{F}_n) = E(H_n X_n|\mathcal{F}_n) = H_n E(X_n|\mathcal{F}_n) \tag{18.2}$$

The last equality holds only if X_n and $X_n H_n$ are integrable. Since we assume that X_n is integrable, $X_n H_n$ is integrable if either H_n is bounded or X_n, H_n are in $\mathcal{L}^2$ (Cauchy Shwartz implies that $X_n H_n$ is integrable in this case).

Martingales in conjunction with stopping times are a neat way of modeling gamblers strategies. A martingale M_n whose differences represent the outcomes at time n may be bet upon by a gambler until he stops at some time. This time is intuitively a stopping time T because (by definition of stopping times) $T = n$ is measurable w.r.t $\mathcal{F}_n$. We can thus define a new process $M_{n \wedge T}$.

Theorem 18.1 *If M_n is an $\mathcal{F}_n-$martingale and T is a stopping time, then $M_{n \wedge T}$ is also an $\mathcal{F}_n-$martingale.*

Proof: Using $H_n = 1_{T > n-1} \in F_{n-1}$ in the MG-transform formula the result is achieved (note that H is bounded). ■

Now, if M_n is a martingale and T is a stopping time bounded by b, then

$$E(M_T) = E(M_{T \wedge b}) = E(M_0) \tag{18.3}$$

In the case of an unbounded stopping time T, we have that $M_{T\wedge n} \to M_T$ a.s. Hence, if expectations and limits can be swapped as in

$$E(lim M_{T\wedge n}) = lim E(M_{T\wedge n}) = E(M_0) \tag{18.4}$$

we can calculate the L.H.S. But, this is not possible always. For instance, in case of a random symmetric walk starting at $S_0 = 1$ and $T = (inf\ n : S_n = 0)$, we have $ES_T = 0$ because $P(T < \infty) = 1$. However, $1 = ES_0 = ES_{T\wedge n}$! As we will see later, uniform integrability is enough to justify the swapping of the expectations and limits.