Characteristic Functions: Inversion Fumula

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circle of ideas around inversion formula for c.f.'s. Ingredients:
1) Parserval identity: Let φ_F, φ_G be the c.f.'s of F, G respectively. Then

$$E(e^{iXY}) = E(\varphi_X(Y)) = \int \varphi_F(y)G(dy)$$

Similarly,

$$E(e^{iXY}) = E(\varphi_Y(X)) = \int \varphi_G(x)F(dx)$$

Thus we get

$$\int \varphi_F(y)G(dy) = \int \varphi_G(x)F(dx)$$

2)There is another interpretation of the integrate $\int f(y)G(dy)$ for suitable f.
Let's recall the convolution. Take X with density $f_X(x)$:

$$P(x \in [a,b]) = \int_a^b f_X(x)dx$$

Take Y with distribution G: $P(Y \leq y) = G(y)$. Then

$$P(X + Y \in dz) = dz \int f_X(z-y)G(dy)$$

i.e. $X + Y$ has density

$$f_{X+Y}(z) = \int f_X(z-y)G(dy)$$

Our key point is to make (2) look as much like (1) as possible. So take $z = 0$ in (2) and pick a symmetric $f, f(y) = f(-y)$. After comparison, we know we need a probability distribution F on $\mathbb{R}$, whose c.f. is symmetric about 0 and also is a probability density up to a constant factor. An ideal choice is Normal. Let Z standard Gaussian. i.e.

$$P(Z \in dz) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$$

It's c.f. is

$$\varphi_Z(t) = e^{-\frac{1}{2}\varphi^2 t^2}$$

So take a general G, if we have known it's c.f., we can recover G by φ_G.
Take F has distribution Z/σ in (1), then

$$\int \varphi_G(x)(\frac{\sigma}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2\sigma^2})dx = \int e^{-\frac{1}{2}y^2/\sigma^2}G(dy)$$

Multiply both sides by $\frac{1}{\sqrt{2\pi}\sigma}$

$$\int \frac{1}{2\pi}\varphi_G(x)e^{-\frac{1}{2}x^2\sigma^2}dx = \int \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}y^2/\sigma^2}G(dy) = f_{Y+\sigma Z}(0)$$

Where Y has the distribution G. This is the thin end of the wedge!
Replace Y with shifted version of Y: $Y = Y - y$, we have

$$f_{Y+\sigma Z}(y) = f_{Y-y+\sigma Z}(0) = \frac{1}{2\pi} \int \varphi_G(x) e^{-ixy - \frac{1}{2}\sigma^2 x^2} dx$$

Under certain conditions, we may get the density of Y by letting $\sigma \to 0$.
Special case:
If $\int |\varphi_G(x)| dx < \infty$, DCT allows us to swap the limit and integrate, then we know Y has a continuous density:

$$f_Y(y) = \frac{1}{2\pi} \int e^{ixy} \varphi_G(x) dx$$