STAT 150 FINAL EXAM, Spring 2006, JWP

Lastname,Firstname and SID#: ______________________________

1. Suppose a Markov chain $(X_n, n = 0, 1, \ldots)$ has transition matrix P. Let i, j, k be states of the chain, and N a positive integer. Derive expressions involving summations and powers of P for the following quantities:

   a) Given $X_0 = i$, the expected number of times n with $1 \le n \le N$ and $X_n = j$.

   b) Given $X_0 = i$, the expected number of times n with $1 \le n \le N$ and $X_n = j$ and $X_{n+1} = k$.

   c) Given $X_0 = i$, the probability that $T_i > 2$, where T_i is the least $n \ge 1$ such that $X_n = i$, and $T_i = \infty$ if there is no such n.

2. Suppose that $(X_1(t), t \ge 0)$ and $(X_2(t), t \ge 0)$ are two independent Markov chains with continuous time parameter, each with state space $\{0, 1\}$, with transition rates λ from 0 to 1 and μ from 1 to 0, for $i = 1, 2$. Let $S_2(t) := X_1(t) + X_2(t)$.

   a) Explain why $(S_2(t), t \ge 0)$ is a Markov chain, and describe its transition rate matrix.

   b) What is the limit distribution of $S_2(t)$ as $t \to \infty$?

   c) Generalize part b) to $S_n(t) := \sum_{i=1}^{n} X_i(t)$ for n independent chains X_i with the same transition rates as before.

3. In a branching process started with one individual, each individual has a number of offspring with uniform distribution on $\{0, 1, \ldots, N\}$ for some fixed $N \ge 1$. Let p_N denote the probability of extinction starting with one individual.

   a) Write down and explain (but do not attempt to solve) an equation solved by p_N.

   b) For each N, indicate how many solutions this equation has in the interval $[0, 1]$. If there is more than one solution, indicate which is p_N.

   c) Find a formula involving p_N for the extinction probability if the branching process starts with a Poisson(λ) number of initial individuals. (Any summation should be simplified.)

4. Cars arrive at a toll booth according to a Poisson process at a long-run rate of 3 cars per minute.

   a) What is the probability that the fourth car arrives within 5 minutes of the second car?

   b) Of the cars arriving at the booth, it has been observed over the long run that 40% carry one person, 25% carry two people, and the rest carry at least three people. What is the probability that in a given 10 minute period exactly 30 cars arrive, with 11 carrying one person, 12 carrying two people, and the other 7 carrying at least three people? State clearly the assumptions you make for this calculation.

5. A Markov chain X with finite state space S and continuous time parameter has the following characteristics. For each state i, the expected holding time in state i before jumping to another state is h_i, and when the jump occurs it is to state $j \neq i$ with probability $p_{i,j}$.

   a) Describe in terms of these parameters h_i and $p_{i,j}$ the matrix A such that the i, j entry of the matrix e^{At} is $P(X_t = j \mid X_0 = i)$.

   b) Assuming that the chain is recurrent, and started with $X_0 = i$, let H_{ij} denote the total amount of time spent in state i up until the first time the chain makes a jump from i to j. Find the distribution of H_{ij}.

6. For a standard Brownian motion B, and $0 < s < t < u$ describe

   a) the conditional distribution of B_t given B_s

   b) the conditional distribution of B_s given B_t and B_u

   c) the conditional distribution of B_t given B_s and B_u

7. For a Poisson process N with rate λ and $0 < s < t < u$ describe

   a) the conditional distribution of N_t given N_s

   b) the conditional distribution of N_s given N_t and N_u

   c) the conditional distribution of N_t given N_s and N_u

8. Find the expected number of Bernoulli(p) trials required to produce the pattern 00110011. Explain your answer in terms of a fair gambling game.

9. Consider a discrete time Markov chain $X_n, n = 0, 1, \ldots$ with state space $\{0, 1, 2, \ldots\}$ and transition matrix P with $P(0, 1) = 1$ and

$$P(i, i+1) = p, \qquad P(i, i-1) = q \text{ for } i \geq 1$$

   for some $0 < p < q < 1$. Let T_0 be the least $n \geq 0$ such that $X_n = 0$. Find

   a) $E(T_0 \mid X_0)$

   b) $E(T_0)$ assuming that the chain is stationary.

10. The transition rates of a continuous time Markov chain on 4 states a, b, c and d are defined as follows by positive parameters α, β, γ and δ.

    From a, the transition rates to b and c both equal α;

    From d, the transition rates to b and c both equal δ;

    From b, the transition rates to a and d both equal β;

    From c, the transition rates to a and d both equal γ;

    All other transition rates between different states are zero.

    a) Display the transition rates in a suitable diagram.

b) Describe the equilibrium distribution of the chain.

c) Starting in state c, let T_c be the first time that the chain returns to state c after having visited some other state. Find $E(T_c)$.