Determinantal Processes And The IID Gaussian Power Series

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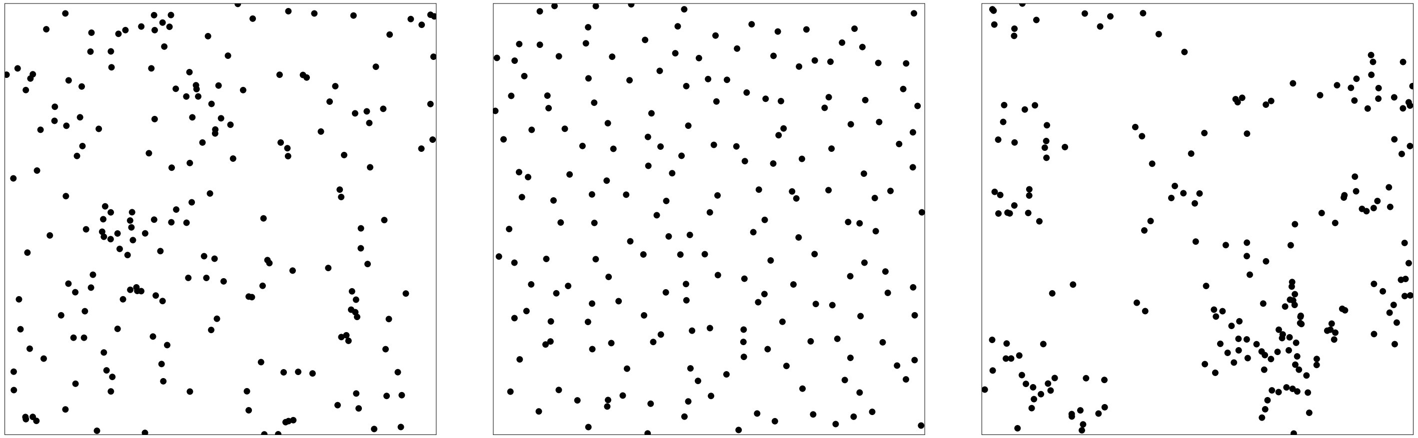

Samples of translation invariant point processes in the plane: Poisson (left), determinantal (center) and permanental for $K(z,w) = \frac{1}{\pi} e^{z\bar{w} - \frac{1}{2}(|z|^2+|w|^2)}$. Determinantal processes exhibit repulsion, while permanental processes exhibit clumping.

Determinantal Point Processes

Let (Ω, μ) be a σ-finite measure space, $\Omega \subset \mathbf{R}^d$. One way to describe the distribution of a point process $\mathcal{X}$ on Ω is via its *joint intensities*.

Definition: $\mathcal{X}$ has **joint intensities** ρ_k, $k = 1, 2, \ldots$ if, for any mutually disjoint (measurable) sets $A_1, \ldots, A_k$,

$$\mathbf{E}\left[\prod_{j=1}^{k} |\mathcal{X} \cap A_j|\right] = \int_{\prod_j A_j} \rho_k(x_1, \ldots, x_k) d\mu$$

In most cases of interest the following is valid (assume no double points):

- Ω *is discrete and* $\mu =$ *counting measure:* $\rho_k(x_1, \ldots, x_k)$ is the probability that $x_1, \ldots, x_k \in \mathcal{X}$.
- Ω *is open in* $\mathbf{R}^d$ *and* $\mu =$ *Lebesgue measure:* $\rho_k(x_1, \ldots, x_k)$ is

$$\lim_{\epsilon \to 0} \frac{\mathbf{P}\left(\mathcal{X} \text{ has a point in each of } B_\epsilon(x_j)\right)}{\left(\mathrm{Vol}(B_\epsilon)\right)^k}.$$

Now let K be the kernel of an integral operator $\mathcal{K}$ on $L^2(\Omega)$ with the spectral decomposition

$$K(x,y) = \sum\nolimits_k \lambda_k \varphi_k(x)\overline{\varphi}_k(y),$$

where $\{\varphi_k\}_k$ is an orthonormal set in $L^2(\Omega)$.

Definition: $\mathcal{X}$ is said to be a **determinantal point process** with kernel K if its joint intensities are

$$\rho_k(x_1, \ldots, x_k) = \det\left((K(x_i, x_j))_{1 \le i,j \le k} \right), \qquad (1)$$

for every $k \geq 1$ and $x_1, \ldots, x_k \in \Omega$.

Key facts: (Maachi)

- A locally finite determinantal process with the Hermitian kernel K exists if and only if K is locally of trace class and $0 \leq \lambda_k \leq 1 \ \forall k$.
- If $K(x, y) = \sum_{k=1}^{n} \varphi_k(x)\overline{\varphi}_k(y)$, then the total number of points in $\mathcal{X}$ is n, almost surely. Since the corresponding integral operator $\mathcal{K}$ on $L^2(\Omega)$ is a projection, such processes are said to be **determinantal projection process**.

Karlin-McGregor (1958)

Consider n independent simple symmetric random walks on $\mathbf{Z}$ started from $i_1 < i_2 < \ldots < i_n$ where all the i_j's are even. Let $P_{i,j}(t)$ be the t-step transition probabilities.

Then the probability that at time t, the random walks are at $j_1 < j_2 < \ldots < j_n$ and have mutually disjoint paths is

$$\det \begin{pmatrix} P_{i_1,j_1}(t) & \ldots & P_{i_1,j_n}(t) \\ \ldots & \ldots & \ldots \\ P_{i_n,j_1}(t) & \ldots & P_{i_n,j_n}(t) \end{pmatrix}.$$

This is intimately related to determinantal processes. For instance, one can show that if t is even and we also condition the walks to return to $i_1, \ldots, i_n$, then the positions of the walks at any time s $(1 \leq s \leq t)$ are determinantal. (See Johanson(2004) for this and more general results)

Uniform Spanning Tree

Let G be a finite undirected graph. Let T be uniformly chosen from the set of spnning trees of G. Orient the edges of G arbitrarily. Let $\check{e}$ be the opposite orientation of e. For each directed edge e, let $\chi^e := \mathbf{1}_e - \mathbf{1}_{\check{e}}$ denote the unit flow along e.

$$\ell^2_-(E) = \{f : E \to \mathbf{R} : f(e) = -f(\check{e})\}$$

$$\star = \text{span}\{\sum_{\underline{e}=v} \chi^e : \text{ where } v \text{ is a vertex.}\}$$

$$\diamondsuit = \text{span}\{\sum_{i=1}^{n} \chi^{e_i} : e_1, \ldots, e_n \text{ is an oriented cycle}\}$$

It is easy to see that $\ell^2_-(E) = \star \oplus \diamondsuit$. Define $I^e := P_\star \chi^e$, the orthogonal projection onto $\star$. Kirchoff (1847) proved that for any edge e, $\mathbf{P}[e \in T] = (I^e, I^e)$.

Theorem: (Burton and Pemantle (1993)) The set of edges in T forms a determinantal process with kernel $Y(e,f) := (I^e, I^f)$. i.e., for any distinct edges $e_1, \ldots, e_k$

$$\mathbf{P}[e_1, \ldots, e_k \in T] = \det\left[(Y(e_i, e_j))_{1\le i,j\le k}\right]. \tag{2}$$

Ginibre Ensemble

Let A be an $n \times n$ matrix with i.i.d. standard complex normal entries. Then the eigenvalues of A form a determinantal process in $\mathcal{C}$ with the kernel

$$K_n(z,w) = \frac{1}{\pi} e^{-\frac{1}{2}(|z|^2+|w|^2)} \sum_{k=0}^{n-1} \frac{(z\overline{w})^k}{k!}.$$

As $n \to \infty$, we get a determinantal process with the kernel

$$\begin{aligned} K(z,w) &= \frac{1}{\pi} e^{-\frac{1}{2}(|z|^2+|w|^2)} \sum_{k=0}^{\infty} \frac{(z\overline{w})^k}{k!}. \\ &= \frac{1}{\pi} e^{-\frac{1}{2}(|z|^2+|w|^2)+z\overline{w}}. \end{aligned}$$

Construction of determinantal projection processes

Define $\mathcal{K}_H \delta_x(\cdot) = K(\cdot, x)$. The intensity measure of the process is given by

$$\mu_H(x) = \rho_1(x) d\mu(x) = \| \mathcal{K}_H \delta_x \|^2 d\mu(x). \tag{3}$$

Note that $\mu_H(M) = \dim(H)$, so $\mu_H / \dim(H)$ is a probability measure on M. We construct the determinantal process as follows. Start with $n = \dim(H)$, and $H_n = H$.

- If $n = 0$, stop.
- Pick a random point X_n from the probability measure μ_{H_n}/n.
- Let $H_{n-1} \subset H_n$ be the orthocomplement of the function $\mathcal{K}_{H_n}\delta_x$ in H_n. In the discrete case, $H_{n-1} = \{f \in H_n : f(X_n) = 0\}$. Note that $\dim(H_{n-1}) = n - 1$ a.s.
- Decrease n by 1 and iterate.

Proposition: The points $(X_1, \ldots, X_n)$ constructed by this algorithm are distributed as a uniform random ordering of the points in a determinantal process $\mathcal{X}$ with kernel K.

Proof: Let $\psi_j = \mathcal{K}_H \delta_{x_j}$. Projecting to H_j is equivalent to first projecting to H and then to H_j, and it is easy to check that $\mathcal{K}_{H_j} \delta_{x_j} = \mathcal{K}_{H_j} \psi_j$. Thus, by (3), the density of the random vector $(X_1, \ldots, X_n)$ constructed by the algorithm equals

$$p(x_1, \ldots, x_n) = \prod_{j=1}^{n} \frac{\| \mathcal{K}_{H_j} \psi_j \|^2}{j}.$$

Note that $H_j = H \cap \langle \psi_{j+1}, \ldots, \psi_n \rangle^{\perp}$, and therefore $V = \prod_{j=1}^{n} \| \mathcal{K}_{H_j} \psi_j \|$ is exactly the repeated "base times height" formula for the volume of the parallelepiped determined by the vectors $\psi_1, \ldots, \psi_n$ in the finite-dimensional vector space $H \subset L^2(M)$. It is well-known that V^2 equals the determinant of the *Gram matrix* whose i, j entry is given by the scalar product of

ψ_i, ψ_j, that is $\int \psi_i \overline{\psi}_j d\mu = K(x_i, x_j)$. We get

$$p(x_1, \ldots, x_n) = \frac{1}{n!} \det(K(x_i, x_j)),$$

so the random variables $X_1, \ldots, X_n$ are exchangeable. Viewed as a point process, the n-point joint intensity of $\{X_j\}_{j=1}^n$ is $n! p(x_1, \ldots, x_n)$, which agrees with that of the determinantal process $\mathcal{X}$. The claim now follows since $\mathcal{X}$ contains n points almost surely.

We have the following remarkable fact that connects the kernel K to the distribution of $\mathcal{X}$:

Theorem: (Shirai-Takahashi (2002)) Suppose $\mathcal{X}$ is a determinantal process on E with kernel $K(x,y) = \sum_k \lambda_k \varphi_k(x)\overline{\varphi}_k(y)$. Then

$$\mathcal{L}(\mathcal{X}) = \sum_{S \subset \mathbf{N}} \alpha(S) \mathcal{L}\left(\mathcal{X}(S)\right), \tag{4}$$

where $\mathcal{X}(S)$ is the determinantal process in E with kernel $\sum_{j \in S} \varphi_j(x)\overline{\varphi}_j(y)$ and

$$\alpha(S) = \prod_{j \in S} \lambda_j \prod_{j \notin S} (1 - \lambda_j).$$

In particular the number of points in the process $\mathcal{X}$ has the distribution of a sum of independent Bernoulli(λ_k) random variables.

Proof: (HKPV) Assume K has finite rank i.e., take

$$K(x,y) = \sum_{k=1}^{n} \lambda_k \varphi_k(x) \overline{\varphi}_k(y).$$

Otherwise we can approximate by finite rank kernels, and deduce the same for general K since the corresponding processes increase (stochastically) to the original process.

Let I_k, $1 \leq k \leq n$ be independent Bernoulli random variables with $I_k \sim \text{Bernoulli}(\lambda_k)$. Then set

$$K_I(x,y) = \sum_{k=1}^{n} I_k \varphi_k(x) \overline{\varphi}_k(y).$$

K_I is a random analogue of the kernel K. We want to prove $\forall m, x_i$'s,

$$\mathbf{E}\left[\det\Big((K_I(x_i,x_j))_{1\leq i,j\leq m}\Big)\right] = \det\Big((K(x_i,x_j))_{1\leq i,j\leq m}\Big). \qquad (5)$$

Proof of (5): Take $m = n$ first. Then we write

$$\begin{pmatrix} K_I(x_1,x_1) & \ldots & \ldots & K_I(x_1,x_n) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ K_I(x_n,x_1) & \ldots & \ldots & K_I(x_n,x_n) \end{pmatrix} =$$

$$\begin{pmatrix} I_1\varphi_1(x_1) & \ldots & I_n\varphi_n(x_1) \\ I_1\varphi_1(x_2) & \ldots & I_n\varphi_n(x_2) \\ \ldots & \ldots & \ldots \\ I_1\varphi_1(x_n) & \ldots & I_n\varphi_n(x_n) \end{pmatrix} \begin{pmatrix} \overline{\varphi}_1(x_1) & \ldots & \overline{\varphi}_1(x_n) \\ \overline{\varphi}_2(x_1) & \ldots & \overline{\varphi}_2(x_n) \\ \ldots & \ldots & \ldots \\ \overline{\varphi}_n(x_1) & \ldots & \overline{\varphi}_n(x_n) \end{pmatrix}.$$

Hence $\det\left((K_I(x_i,x_j))\right)_{1\le i,j\le n} = I_1 \ldots I_n \det(A^*A)$ where A is the second matrix on the right side above. On taking expectations we get

$$\mathbf{E}\left[\det\left((K_I(x_i,x_j))_{1\le i,j\le n}\right)\right] = \lambda_1 \ldots \lambda_n \det(A^*A).$$

Now we also have

$$\begin{pmatrix} K(x_1,x_1) & \dots & \dots & K(x_1,x_n) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K(x_n,x_1) & \dots & \dots & K(x_n,x_n) \end{pmatrix} =$$

$$\begin{pmatrix} \lambda_1\varphi_1(x_1) & \dots & \lambda_n\varphi_n(x_1) \\ \lambda_1\varphi_1(x_2) & \dots & \lambda_n\varphi_n(x_2) \\ \dots & \dots & \dots \\ \lambda_1\varphi_1(x_n) & \dots & \lambda_n\varphi_n(x_n) \end{pmatrix} \begin{pmatrix} \overline{\varphi}_1(x_1) & \dots & \overline{\varphi}_1(x_n) \\ \overline{\varphi}_2(x_1) & \dots & \overline{\varphi}_2(x_n) \\ \dots & \dots & \dots \\ \overline{\varphi}_n(x_1) & \dots & \overline{\varphi}_n(x_n) \end{pmatrix}.$$

From this we get

$$\det\Big((K(x_i,x_j))_{1\le i,j\le n}\Big) = \lambda_1 \dots \lambda_n \det(A^*A).$$

This proves that the two point processes $\mathcal{X}$ (determinantal with kernel K) and $\mathcal{X}_I$ (determinantal with kernel K_I) have the same n-point joint intensity.

But both these processes have at most n points. Therefore for every m, the m-point joint intensities are determined by the n-point joint intensities (zero for $m > n$, got by integrating for $m < n$).

This proves the theorem.

Zeros of the *i.i.d.* Gaussian power series [Virág-P.].

Let

$$f_U(z) = \sum_{n=0}^{\infty} a_n z^n$$
$$Z_U = \text{zeros}(f_U) \qquad (6)$$

with $\{a_n\}$ complex Gaussian, density$(re^{i\theta}) = e^{-r^2}$.

Theorem: (Hannay, Zelditch-Shiffman, ...)
Law of $\mathbf{Z}_U$ invariant under Möbius transformations $z \to e^{i\alpha}\frac{z-\lambda}{1-\bar{\lambda}z}$ that preserve unit disk.

Euclidean analog:

$$f_{\mathcal{C}} = \sum_{n=0}^{\infty} \frac{a_n z^n}{\sqrt{n!}}, \tag{7}$$

satisfies

$$\begin{aligned}\mathrm{Cov}[f_{\mathcal{C}}(z), f_{\mathcal{C}}(w)] &= \mathbf{E}\left[\textstyle\sum_n \frac{a_n z^n}{\sqrt{n!}} \cdot \sum_k \frac{\bar{a}_k \bar{w}^k}{\sqrt{k!}}\right] \\ &= \textstyle\sum_{n=0}^{\infty} \frac{z^n \bar{w}^n}{n!} = e^{z\bar{w}}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{Cov}[f_{\mathcal{C}}(z+a), f_{\mathcal{C}}(w+a)] &= e^{(z+a)(\bar{w}+\bar{a})} \\ &= \mathrm{Cov}\left[e^{|a|^2/2} e^{\bar{a}z} f_{\mathcal{C}}(z), e^{|a|^2/2} e^{\bar{a}w} f_{\mathcal{C}}(w)\right].\end{aligned}$$

Since Gaussian processes are determined by $\mathrm{Cov}(\cdot, \cdot)$ this proves translation invariance of $\mathrm{Law}[\mathrm{zeros}(f_{\mathcal{C}})]$.

Definition: Let $p_\epsilon(z_1, \ldots, z_n)$ denote the probability that a random function f has zeros in $B_\epsilon(z_1), \ldots B_\epsilon(z_n)$. **Joint intensity** of zeros (if it exists) is defined to be

$$p(z_1, \ldots, z_n) = \lim_{\epsilon \downarrow 0} \frac{p_\epsilon(z_1, \ldots, z_n)}{(\pi\epsilon^2)^n} \tag{8}$$

Theorem: (Hammersley)
Let f be a Gaussian analytic function in a planar domain D, $z_1, \ldots, z_n \in D$, and consider the matrix $A = \left(\mathbf{E} f(z_i)\overline{f(z_j)}\right)$. If A is non-singular then $p(z_1, \ldots z_n)$ exists and equals

$$\frac{\mathbf{E}\left(|f'(z_1) \cdots f'(z_n)|^2 \,\middle|\, f(z_1) = \cdots = f(z_n) = 0 \right)}{\det(\pi A)}.$$

Theorem: (Virág - P.)
The joint intensity of zeros for f_U is

$$\begin{aligned} p(z_1,\ldots,z_n) &= \pi^{-n}\det\left[\frac{1}{(1-z_i\bar{z_j})^2}\right]_{i,j} \\ &= \det[K(z_i,z_j)] \end{aligned}$$

where $K(z,w)=\frac{1}{\pi(1-z\bar{w})^2}$ is the Bergman kernel for U.

Proof of Determinantal Formula

Let

$$T_\beta(z) = \frac{z - \beta}{1 - \overline{\beta} z} \tag{9}$$

denote a Möbius transformation fixing the unit disk. Also, for fixed $z_1, \ldots, z_n \in \mathbf{U}$ denote

$$\Upsilon(z) = \prod_{j=1}^{n} T_{z_j}(z). \tag{10}$$

Key facts:

1. Let$f = f_U$ and $z_1, \ldots, z_n \in \mathbf{U}$. The distribution of the random function $T_{z_1}(z) \cdots T_{z_n}(z) f(z)$ is the same as the conditional distribution of $f(z)$ given $f(z_1) = \ldots = f(z_n) = 0$.

2. It follows that the conditional joint distribution of the random variables $\big(f'(z_k) \,:\, k = 1, \ldots, n\big)$ given $f(z_1) = \ldots = f(z_n) = 0$, is the same as the unconditional joint distribution of $\big(\Upsilon'(z_k) f(z_k) \,:\, k = 1, \ldots, n\big)$.

3. Consider the $n \times n$ matrices

$$\begin{aligned} A_{jk} &= \mathbf{E} f(z_j)\overline{f(z_k)} = (1 - z_j \overline{z}_k)^{-1}, \\ M_{jk} &= (1 - z_j \overline{z}_k)^{-2}. \end{aligned}$$

By the classical Cauchy determinant formula,

$$\begin{aligned} \det(A) &= \prod_{k,j} \frac{1}{1 - \overline{z}_j z_k} \prod_{k<j} (z_k - z_j)(\overline{z}_k - \overline{z}_j) \\ &= \prod_{k=1}^{n} |\Upsilon'(z_k)|. \qquad (11) \end{aligned}$$

4. We also use Borchardt's identity:

$$\operatorname{perm}\left(\frac{1}{x_j+y_k}\right)_{j,k}\det\left(\frac{1}{x_j+y_k}\right)_{j,k}=\det\left(\frac{1}{(x_j+y_k)^2}\right)_{j,k}$$

setting $x_j = z_j^{-1}$ and $y_k = -\overline{z_k}$ and dividing both sides by $\prod_j z_j^2$, gives that

$$\operatorname{perm}(A)\det(A) = \det(M). \tag{12}$$

5. Finally, recall the Gaussian moment formula: If $Z_1, \ldots, Z_n$ are jointly complex Gaussian random variables with covariance matrix $C_{jk} = \mathbf{E}Z_j\overline{Z_k}$, then $\mathbf{E}\big(|Z_1 \cdots Z_n|^2\big) = \operatorname{perm}(\mathcal{C})$.

From Hammersley's formula $p(z_1, \ldots, z_n)$ equals

$$\frac{\mathbf{E}\Big(|f'(z_1)\cdots f'(z_n)|^2 \,\Big|\, f(z_1),\ldots,f(z_n)=0\Big)}{\pi^n \det(A)} \,.$$

The numerator equals

$$\mathbf{E}\big(|f(z_1)\cdots f(z_n)|^2\big) \, \prod_k |\Upsilon'(z_k)|^2 = \mathrm{perm}(A) \det(A)^2 \,,$$

where the last equality uses the Gaussian moment formula. Thus,

$$\begin{aligned} p(z_1, \ldots, z_n) &= \pi^{-n} \mathrm{perm}(A) \det(A) \\ &= \pi^{-n} \det(M). \end{aligned}$$

Theorem 2: (Virág - P.)
Let

$$X_k \sim \begin{cases} 1 & r^{2k} \\ 0 & 1 - r^{2k} \end{cases}$$

be independent. Then $\sum_1^\infty X_k$ and $N_r = |\mathbf{Z}_U \cap B(0,r)|$ have same distribution.

Corollary: Let $h_r = 4\pi r^2/(1-r^2)$ (hyperbolic area). Then

$$\mathbf{P}(N_r = 0) = e^{-h_r \frac{\pi}{24} + o(h_r)} = e^{\frac{-\pi^2/12 + o(1)}{1-r}}. \tag{13}$$

All of the above generalize to other simply connected domains with smooth boundary.

$$\mathbf{E}\left(f_D(z)\overline{f_D(w)}\right) = 2\pi S_D(z,w) \text{ (Szëgo Kernel)} \tag{14}$$

Denote $q = r^2$. Key to law of $N_r = |\mathbf{Z}_U \cap B(0,r)|$:

$$
\begin{aligned}
\mathbf{E}\binom{N_r}{k} &= \frac{1}{k!}\int_{B_r^k} p(z_1,\ldots,z_k)dz_1,\ldots dz_k \\
&= \frac{q^{\binom{k+1}{2}}}{(1-q)(1-q^2)\ldots(1-q^k)} \\
&= \gamma_k.
\end{aligned}
$$

Euler's partition identity

$$
\sum_{k=0}^{\infty} \gamma_k s^k = \prod (1+q^k s), \tag{15}
$$

implies that

$$
\mathbf{E}(1+s)^{N_r} = \sum_{k=0}^{\infty} \mathbf{E}\binom{N_r}{k} s^k = \sum \gamma_k s^k \tag{16}
$$

has product form!

Dynamics

Let

$$f_U(t,z) = \sum_n a_n(t) z^n \tag{17}$$

with $a_n(t)$ performing Ornstein-Uhlenbeck diffusion, $a_n(t) = e^{-t/2} W_n(e^t)$. Suppose that the zero set of f_U contains the origin. Movement of this zero satisfies stochastic differential equation

$$dz = \sigma dW \tag{18}$$

where

$$\frac{1}{\sigma} = |f_U'(0)| = c \lim_{r \uparrow 1} \frac{1}{\sqrt{1-r^2}} \prod_{\substack{z \in Z_U \\ 0<|z|<r}} |z| = \tilde{c} \prod_{k=1}^{\infty} e^{1/k} |z_k|. \tag{19}$$