Point processes, the stable marriage algorithm and Gaussian power series

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A **point process** is a random set of points in space; determined by its counting function

$$N(A) = \#\{\text{points of process in } A\}.$$

We focus on processes with isometry-invariant distribution.

Examples:

1. Poisson point process (with intensity 1) $N(A_i)$ independent for disjoint A_i.

$$\mathbf{P}\left[N(A_i) = k\right] = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \lambda = \mathcal{L}(A_k).$$

2. $\mathbf{Z}^d + U, \quad U \sim \text{Unif}[0,1]^d$

These examples can be decorated:

(a) $\mathbf{Z}^d + F + U$, F finite, $U \sim \text{Unif}[0,1]^d$

(b) Remove every point of $\mathbf{Z}^d$ with probability q, independently then add U.

(c) $\{m + X_m + U : m \in \mathbf{Z}^d\}$ with $\{X_m\}$ *i.i.d.* .

(d) Poisson + F

Fundamentally different:

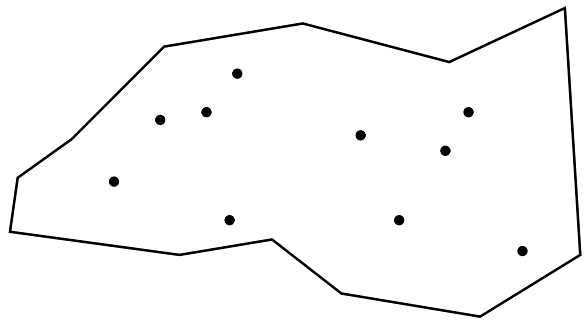
Zeros of

$$f_{\mathcal{C}} = \sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{n!}} \tag{1}$$

where $a_n \sim$ complex Gaussian, independent identical law.

Land division problem:

Given n “farmers” in a planar region (of area n) partition region into n equal parts (in area) by a “local” method (without “central planning”)



Math problem:

Given point process with intensity one (expect 1 point per unit area) allocate to each point a unit of area in invariant way.

Gale - Shapley stable marriage problem (1962)
Given n men and n women with preference orderings, a perfect matching is **unstable** if there are a man and a woman that prefer each other to current mates.

Then: (Gale-Shapley)
A stable matching always exists (it may not be unique).

Note: Existence can fail in the one-gender version.

Gale Shapley stable marriage algorithm

1. Each man proposes to preferred woman.
2. Each woman rejects all but top proposer.
3. Rejected men propose to next woman on their list.
4. Repeat steps 2,3 until each woman has unique man proposing to her.

Theorem (Gale-Shapley)

* Resulting matching is stable.
* Stable matching is unique if and only if men-proposing algorithm and women-proposing algorithm both yield the same matching.

Many - to - one allocation

The Gale-Shapley algorithm is used to assign residents to hospitals. In our geometric setting, both “centers” (points of point process) and points of plane use distance to order.

- Each center “wants” to be allotted one unit of area, as close as possible.
- Each point in plane “wants” to be assigned to a close center.
- Existence provided by Gale-Shapley algorithm and ergodicity. In this case uniqueness also holds (a.s.)

Claim: At appetite 1 and intensity 1, all centers are sated and unallocated points have "area" = 0, with probability 1.
(This uses invariance: it fails for $\mathbf{Z}^2 - [0, 10]^2$).

Proof of Claim: For $u, v \in \mathbf{Z}^d$ define $Q_u = u + [0, 1]^d$ and $Q_v = v + [0, 1]^d$. Let $m(u, v)$ be the expected area in Q_u allocated to centers in Q_v.

$$(*) \quad \sum_v m(u, v) = \sum_v m(u, u + v) = \sum_v m(u - v, u) = \sum_v m(v, u)$$

Instance of "Mass transport principle". If
$(*) < 1$ then there exist unsated centers and unallocated area, a contradiction. Note, by ergodicity, $\mathbf{P}(\exists \text{ unsated centers}) \in \{0, 1\}$ and $\mathbf{P}(\exists \text{ unallocated area}) \in \{0, 1\}$.

In general, stable matchings need not be unique.

$$\begin{array}{ccccc} 1 & \rightarrow & A & & 1 \\ & & & \nwarrow\!\!\!\!\nearrow\!\!\!\!\searrow & \\ 2 & \rightarrow & B & & 2 \\ \text{men} & & \text{women} & & \text{men} \end{array}$$

Why does uniqueness hold for our stable allocations?

Simpler setting: Given n men and n women in a metric space, with preference by distance (close is preferred), there exists a stable matching. Both men-proposing and women-proposing algorithms minimize

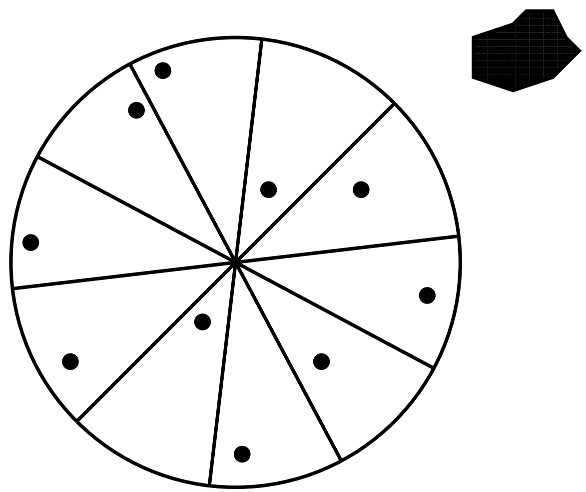
$$\sum_{x} d(x, \pi(x)) = \sum_{x \sim y} d(x, y) \tag{2}$$

over stable matchings.

“Soft” ergodic theory argument shows that for appetite = 1 a.s. every center is sated and all points of plane (up to area zero) are allocated. (same in space and in hyperbolic plane).

Theorem: In $\mathbf{R}^d$, the region allocated to each center is a.s. bounded.

Proof: Sector argument for hungry centers. Hungry centers (if exist) form ergodic point process. In figure below, can hungry center in middle be allocated the shaded region to the right?



Thus if θ is center closest to origin and R_θ radius of its region, then

$$\mathbf{P}(R_\theta > r) \to 0 \text{ as } r \to \infty \tag{3}$$

Open problem: Get rates. For $d = 1$ we do have lower bounds.

- In plane, $\mathbf{E}R_\theta = \infty$

- For $d > 2$, $\mathbf{E}R_\theta^d = \infty$

but no upper bounds for $d > 1$.

Extra head rules

Definition: Let $\{X_r\}_{r\in\mathbf{Z}^d}$ be *i.i.d.* with

$$X_r \sim \begin{cases} 1 & p \\ 0 & 1-p \end{cases}.$$

An **extra head rule** is a random variable τ ($\mathbf{Z}^d$-valued) such that

1. $\mathcal{X}_\tau = 1$ a.s.
2. $\{X_{\tau+\nu}\}_{\nu\in\mathbf{Z}^d\setminus 0}$ are *i.i.d.* $\sim \begin{cases} 1 & p \\ 0 & 1-p \end{cases}.$

If $\tau = \tau(X)$, we say that τ is a **non-randomized** extra head rule.

* Thorisson (1996) proved that extra head rules exist in great generality.
* Liggett (2002) constructed explicit examples of non-randomized rules. He asked when such rules exist.

Constructing extra head rules

* First, a construction that **fails**. Take $p = 1/2$ on $\mathbf{Z}$, choose the first "one" to the right of the origin. Observe that

$$\mathbf{P}(X_{\tau-1} = 1) = \tfrac{1}{4}$$

for $\tau = \min\{k \geq 0 : X_k = 1\}$.

* Liggett proved that

$$\tau = \min\{k \geq 0 : X_0 + x_1 + \ldots + X_k \geq k/2\} \tag{4}$$

works.

* More generally, if $p = \mathbf{E}X_i = 1/\ell$, then

$$\tau = \min\left\{k \geq 0 : \sum_{i=0}^{k}\left(X_i - \frac{1}{\ell}\right) \geq 0\right\} \tag{5}$$

works, but only when ℓ is an **integer**.

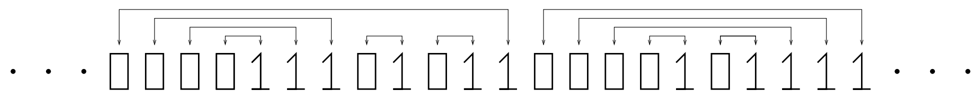
Theorem (Holroyd - P.):
In $\mathbf{Z}^d$ (and in any discrete group G) a non-randomized extra head rule exists iff $1/p \in \mathbf{Z}$.

There exists a 1-1 correspondence between such rules and **invariant allocations** $\Psi_X : G \to G$ with

$$|\Psi_X^{-1}(v)| = \begin{cases} 0 & X_v = 0 \\ 1/p & X_v = 1 \end{cases} \tag{6}$$

Example:

For $p = 1/2$, an invariant matching (of zeros and ones) yields an invariant allocation.



This matching corresponds to the allocation $\Psi(k) = k$ for $X(k) = 1$ and

$$\Psi(k) = \min\left\{ n \geq k : \sum_{j=k}^{n} (-1)^{X_j} \leq 0 \right\} \tag{7}$$

for $X(k) = 0$. Liggett's rule $\tau = \Psi(0)$ then gives an extra heads rule. This occurred earlier in work of Meshalkin (1960) on finitary coding.

Proof that invariant allocation implies extra head rule

Let A_k be the event that

$$\begin{array}{cccccc} \dots & 1 & * & 1 & 0 & \dots \\ \dots & \uparrow & \uparrow & \uparrow & \uparrow & \dots \\ \dots & k-1 & k & k+1 & k+2 & \dots \end{array}$$

Then compute:

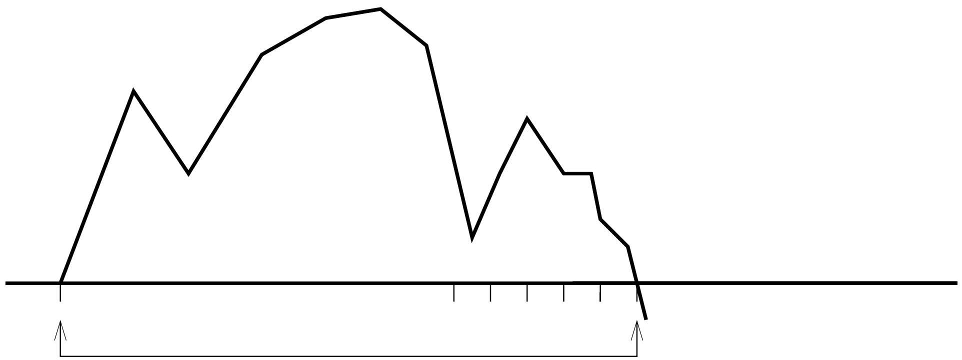
$$\begin{aligned} \mathbf{P}(A_{\Psi(0)}) &= \sum_{k=-\infty}^{\infty} \mathbf{P}(\Psi(0) = k, A_k) \\ &= \sum_{k=-\infty}^{\infty} \mathbf{P}(\Psi(-k) = 0, A_0) \\ &= \mathbf{E}\left[\#\Psi^{-1}(0) \cdot \mathbf{1}_{A_0} \cdot \mathbf{1}_{X(0)=1}\right] \\ &= p^{-1}\mathbf{P}(A(0))p \end{aligned}$$

“Extra heads” for point processes

Theorem (Thorisson):
Given shift-invariant point process in $\mathbf{R}^d$ (and more generally) there is a (possibly randomized) rule to choose a point of the process from which we will see the palm measure. In particular, for Poisson, removing that point after shifting it to the origin will yield back the Poisson process.

Is there a non-randomized extra head rule in continuum?

- Yes for $d = 1$ by version of Liggett's rule:



- In lattice, an extra head rule on $\mathbf{Z}$ immediately gives one for $\mathbf{Z}^d$ (work line by line). This fails in continuum...

- **Theorem** (Holroyd - P.):
  For invariant point processes extra head rules exists iff invariant allocations exist.

Large Deviations

Theorem: Consider Poisson(1) centers in $\mathbf{R}^d$ and suppose each center has appetite $\rho > 1$. Then origin is allocated to center $\Psi(0)$ by stable allocation with

$$\mathbf{P}\left[|\Psi(0)| > R\right] \leq \exp\left[-C(\rho)R^d\right] \qquad (8)$$

and $C(\rho) > 0$ for $\rho > 1$. But we have no estimate for $C(\rho)$ if $\rho \sim 1$ and $d > 1$.

Easy case: Appetite $\rho > 4$ in plane. Centers in $B(0, R)$ prefer origin to points outside $B(0, 2R)$.

$$\begin{aligned} \mathbf{P}\left[|\Psi(0)| > R\right] &\leq \mathbf{P}\left[\#\text{ centers in } B(0,R) \leq \frac{4\pi R^2}{\rho}\right] \\ &\sim \exp\left[-\frac{(\rho-4)\pi}{2\rho}R^2\right] \qquad (9) \end{aligned}$$

Determinantal Point Processes

Let (E, μ) is a σ-finite measure space, $E \subset \mathbf{R}^d$. One way to describe the distribution of a point process $\mathcal{X}$ on E is via its' *Correlation functions*.

Definition: $\mathcal{X}$ has correlations ρ_k, $k = 1, 2, \ldots$ if, for any m and any $k_1, \ldots, k_m \geq 0$, and any mutually disjoint (measurable) $A_1, \ldots, A_m$,

$$\mathbf{E}\left[\prod_{j=1}^{m} \frac{(|\mathcal{X} \cap A_j|)!}{(|\mathcal{X} \cap A_j| - k_j)!}\right] = \int_{\prod_j A_j^{k_j}} \rho_k(x_1, \ldots, x_k).$$

where $k = k_1 + \ldots + k_m$.

In most cases of interest the following is valid (assume no double points)-

- *E is discrete and $\mu =$ counting measure:* $\rho_k(x_1, \ldots, x_k)$ is the probability that $x_1, \ldots, x_k \in \mathcal{X}$.

- *E is open in $\mathbf{R}^d$ and $\mu =$ Lebesgue measure:* $\rho_k(x_1, \ldots, x_k)$ is

$$\lim_{\epsilon \to 0} \frac{\mathbf{P}\left(\mathcal{X} \text{ has a point in each of } B_\epsilon(x_j)\right)}{\left(\mathrm{Vol}(B_\epsilon)\right)^k}.$$

Now let K be an integral operator on E with the spectral decomposition

$$K(x,y) = \sum_k \lambda_k \varphi_k(x)\overline{\varphi}_k(y),$$

where $\{\varphi_k\}_k$ is an orthonormal set in $L^2(E)$.

Definition $\mathcal{X}$ is said to be a determinantal point process with kernel K if its correlations are

$$\rho_k(x_1, \ldots, x_k) = \det\left((K(x_i, x_j))\right)_{1\le i,j\le k}, \tag{10}$$

for every $k \geq 1$ and $x_1, \ldots, x_k \in E$.

Key facts:

- A locally finite determinantal process with the Hermitian kernel K exists if and only if K is locally of trace class and $0 \leq \lambda_k \leq 1\ \forall k$.
- If $K(x,y) = \sum_{k=1}^{n} \varphi_k(x)\overline{\varphi}_k(y)$, then the total number of points in $\mathcal{X}$ is n, almost surely.
- If $K \leq L$ in the sense of operators, then $\mathcal{X}_K \leq \mathcal{X}_L$ in the sense of stochastic domination.

We have the following remarkable fact that connects the kernel K to the distribution of $\mathcal{X}$:

Theorem: Suppose $\mathcal{X}$ is a determinantal process on E with kernel $K(x,y) = \sum_k \lambda_k \varphi_k(x)\overline{\varphi}_k(y)$. Then

$$\mathcal{L}(\mathcal{X}) = \sum_{S \subset \mathbf{N}} \alpha(S) \mathcal{L}\left(\mathcal{X}(S)\right), \tag{11}$$

where $\mathcal{X}(S)$ is the determinantal process in E with kernel $\sum_{j \in S} \varphi_j(x)\overline{\varphi}_j(y)$ and

$$\alpha(S) = \prod_{j \in S} \lambda_j \prod_{j \notin S} (1 - \lambda_j).$$

In particular the number of points in the process $\mathcal{X}$ has the distribution of a sum of independent Bernoulli(λ_k) random variables.

Proof: Assume K has finite rank i.e., take

$$K(x,y) = \sum_{k=1}^{n} \lambda_k \varphi_k(x) \overline{\varphi}_k(y).$$

Otherwise we can approximate by finite rank kernels, and deduce the same for general K since the corresponding processes increase (stochastically) to the original process.

Let I_k, $1 \le k \le n$ be independent Bernoulli random variables with $I_k \sim \text{Bernoulli}(\lambda_k)$. Then set

$$K_I(x,y) = \sum_{k=1}^{n} I_k \varphi_k(x) \overline{\varphi}_k(y).$$

K_I is a random analogue of the kernel K. We want to prove $\forall m, x_i$s,

$$\mathbf{E}\left[\det\left((K_I(x_i,x_j))\right)_{1\le i,j\le m}\right] = \det\left((K(x_i,x_j))\right)_{1\le i,j\le m}. \qquad (12)$$

Proof of (12): Take $m = n$ first. Then we write

$$\begin{pmatrix} K_I(x_1,x_1) & \dots & \dots & K_I(x_1,x_n) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K_I(x_n,x_1) & \dots & \dots & K_I(x_n,x_n) \end{pmatrix} =$$

$$\begin{pmatrix} I_1\varphi_1(x_1) & \dots & I_n\varphi_n(x_1) \\ I_1\varphi_1(x_2) & \dots & I_n\varphi_n(x_2) \\ \dots & \dots & \dots \\ I_1\varphi_1(x_n) & \dots & I_n\varphi_n(x_n) \end{pmatrix} \begin{pmatrix} \overline{\varphi}_1(x_1) & \dots & \overline{\varphi}_1(x_n) \\ \overline{\varphi}_2(x_1) & \dots & \overline{\varphi}_2(x_n) \\ \dots & \dots & \dots \\ \overline{\varphi}_n(x_1) & \dots & \overline{\varphi}_n(x_n) \end{pmatrix}.$$

Hence $\det\left((K_I(x_i,x_j))\right)_{1\le i,j\le n} = I_1 \dots I_n \det(A^*A)$ where A is the second matrix on the right side above. On taking expectations we get

$$\mathbf{E}\left[\det\left((K_I(x_i,x_j))\right)_{1\le i,j\le n}\right] = \lambda_1 \dots \lambda_n \det(A^*A).$$

Now we also have

$$\begin{pmatrix} K(x_1,x_1) & \dots & \dots & K(x_1,x_n) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K(x_n,x_1) & \dots & \dots & K(x_n,x_n) \end{pmatrix} =$$

$$\begin{pmatrix} \lambda_1\varphi_1(x_1) & \dots & \lambda_n\varphi_n(x_1) \\ \lambda_1\varphi_1(x_2) & \dots & \lambda_n\varphi_n(x_2) \\ \dots & \dots & \dots \\ \lambda_1\varphi_1(x_n) & \dots & \lambda_n\varphi_n(x_n) \end{pmatrix} \begin{pmatrix} \overline{\varphi}_1(x_1) & \dots & \overline{\varphi}_1(x_n) \\ \overline{\varphi}_2(x_1) & \dots & \overline{\varphi}_2(x_n) \\ \dots & \dots & \dots \\ \overline{\varphi}_n(x_1) & \dots & \overline{\varphi}_n(x_n) \end{pmatrix}.$$

From this we get

$$\det\left((K(x_i,x_j))\right)_{1\le i,j\le n} = \lambda_1 \dots \lambda_n \det(A^*A).$$

This proves that the two point processes $\mathcal{X}$ (determinantal with kernel K) and $\mathcal{X}_I$ (determinantal with kernel K_I) have the same n-point correlation.

But both these processes have at most n points. Therefore for every m, the m-point correlations are determined by the n-point correlation (zero for $m > n$, got by integrating for $m < n$).

This proves the theorem.

Karlin-McGregor (1958)

Consider n independent simple symmetric random walks on $\mathbf{Z}$ started from $i_1 < i_2 < \ldots < i_n$ where all the i_j's are even. Let $P_{i,j}(t)$ be the t-step transition probabilities.

Then the probability that at time t, the random walks are at $j_1 < j_2 < \ldots < j_n$ and have mutually disjoint paths is

$$\det \begin{pmatrix} P_{i_1,j_1}(t) & \ldots & P_{i_1,j_n}(t) \\ \ldots & \ldots & \ldots \\ P_{i_n,j_1}(t) & \ldots & P_{i_n,j_n}(t) \end{pmatrix}.$$

This is intimately related to determinantal processes. For instance, one can show that if t is even and we also condition the walks to return to $i_1, \ldots, i_n$, then, the positions of the random walk at any epoch s $(1 \le s \le t)$ is determinantal. (See Johanson(2004) for this and more general results)

Uniform Spanning Tree

Let G be a finite undirected graph. Let T be uniformly chosen from the set of spnning trees of G. Orient the edges of G arbitrarily. Let $\check{e}$ be the opposite orientation of e. For each directed edge e, let $\chi^e := \mathbf{1}_e - \mathbf{1}_{\check{e}}$ denote the unit flow along e.

$$\ell^2_-(E) = \{f : E \to \mathbf{R} : f(e) = -f(\check{e})\}$$

$$\star = \text{span}\{\sum_{\underline{e}=v} \chi^e : \text{ where } v \text{ is a vertex.}\}$$

$$\diamondsuit = \text{span}\{\sum_{i=1}^{n} \chi^{e_i} : e_1, \ldots, e_n \text{ is an oriented cycle}\}$$

It is easy to see that $\ell^2_-(E) = \star \oplus \diamondsuit$.

Define $I^e := P_\star \chi^e$, the orthogonal projection onto $\star$. Kirchoff (1847) proved that for any edge e, $\mathbf{P}[e \in T] = (I^e, I^e)$.

Theorem: (Burton and Pemantle (1993)) The set of edges in T forms a determinantal process with kernel $Y(e, f) := (I^e, I^f)$. i.e., for any distinct edges $e_1, \ldots, e_k$

$$\mathbf{P}[e_1, \ldots, e_k \in T] = \det[Y(e_i, e_j)]_{1 \le i,j \le k}.$$

Proof: (Benjamini-Lyons-Schramm-P. (2001))
If some cycle can be formed from the edges $e_1, \ldots, e_k$, then a linear combination of the corresponding columns of $[Y(e_i, e_j)]$ is zero: suppose that such a cycle is $\sum_j a_j \chi^{e_j} \in \diamondsuit$, where $a_j \in \{-1, 0, 1\}$. Then

$$\sum_j a_j Y(e_i, e_j) = \sum_j a_j (I^{e_i}, \chi^{e_j}) = \left(I^{e_i}, \sum_j a_j \chi^{e_j} \right) = 0$$

because $I^{e_i} \in \star$ and $\star$ is orthogonal to $\diamondsuit$. Thus both sides vanish.

Now onwards we assume that there are no such cycles. Since $[Y(e_i, e_j)]_{1\le i,j\le k}$ is the Gram matrix of $I^{e_1}, \ldots, I^{e_k}$, $\det[Y(e_i, e_j)]_{1\le i,j\le k}$ is the absolute square of the volume of the parallelepiped formed by these vectors. This volume can also be computed as

$$\prod_{i=1}^{k} \left| P_{Z_i}^{\perp} I^{e_i} \right|^2 \tag{13}$$

where Z_i is the linear span of $I^{e_1}, \ldots, I^{e_{i-1}}$.

Claim: $P_{Z_i}^{\perp} I^{e_i} = \widehat{I}^{e_i}(e_i)$ where $\widehat{I}$ denotes the current in the graph $G/e_1, \ldots, e_{i-1}$ obtained from G by contracting the edges $e_1, \ldots, e_{i-1}$.

Proof of the Claim: Let $\widehat{\star}$ and $\widehat{\Diamond}$ be the star and cycle spaces in $G/e_1, \ldots, e_{i-1}$.

$$\begin{aligned} \star \cap \widehat{\Diamond} &= P_\star \widehat{\Diamond} \qquad\qquad (\text{because } \star \supset \widehat{\star}, \Diamond \subset \widehat{\Diamond}) \\ &= P_\star \Diamond + P_\star \operatorname{span}\langle \chi^{e_1}, \ldots, \chi^{e_{i-1}} \rangle \\ &= P_\star \operatorname{span}\langle \chi^{e_1}, \ldots, \chi^{e_{i-1}} \rangle \\ &= Z_i. \end{aligned}$$

Therefore

$$\begin{aligned} \ell^2_-(E) &= \widehat{\star} \oplus \widehat{\Diamond} \\ &= \widehat{\star} \oplus \widehat{\Diamond} \cap \star \oplus \widehat{\Diamond} \cap \Diamond \\ &= \widehat{\star} \oplus Z_i \oplus \Diamond. \end{aligned}$$

This shows that $P_{Z_i}^{\perp} I^{e_i} = P_{\widehat{\star}} \chi^{e_i} = \widehat{I}^{e_i}(e_i)$ as claimed.

We have thus shown

$$\det[Y(e_i, e_j)]_{1 \le i,j \le k} = \prod_{i=1}^{k} \left(\widehat{I}_i^{e_i}, \widehat{I}_i^{e_i} \right)$$

where $I_i^{e_i}$ is computed in $G/e_1, \ldots, e_{i-1}$. By Kirchoff's theorem, this is the probability that e_i belongs to the uniform spanning tree on $G/e_1, \ldots, e_{i-1}$.

The UST on $G/e_1, \ldots, e_{i-1}$ has the same distribution as the UST on G conditioned to contain $e_1, \ldots, e_{i-1}$. Thus

$$\begin{aligned} \det[Y(e_i, e_j)]_{1 \le i,j \le k} &= \prod_{i=1}^{k} \mathbf{P}[e_i \in T \mid e_k, \in T, k < i] \\ &= \mathbf{P}[e_1, \ldots, e_i \in T]. \end{aligned}$$

Gaussian Unitary Ensemble

Let A be an $n \times n$ matrix with i.i.d. $CN(0,1)$ entries. The eigenvalues of $\frac{(A+A^*)}{2}$ form a determinantal process on $\mathbf{R}$ with kernel

$$K_n(x,y) = \sum\nolimits_{k=0}^{n-1} \varphi_k(x)\varphi_k(y),$$

where, $\varphi_k(x)$ are the Hermite functions:

$$\varphi_k(x) := \frac{1}{\sqrt{2^k k! \sqrt{\pi}}} e^{x^2/2} \left(\frac{-d}{dx} \right)^k e^{-x^2}.$$

As $n \to \infty$, $K_n(x,y)$ diverges (intensity of point processes $\to \infty$). Rescaling the line by $\frac{\sqrt{2N}}{\pi}$ gives a determinantal process on $\mathbf{R}$ with kernel

$$K(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}.$$

Since K is a function of $x-y$, the distribution of the process is translation invariant.

Ginibre Ensemble

Let A be an $n \times n$ matrix with i.i.d. standard complex normal entries. Then the eigenvalues of A form a determinantal process in $\mathcal{C}$ with the kernel

$$K_n(z,w) = \frac{1}{\pi} e^{-\frac{1}{2}(|z|^2+|w|^2)} \sum_{k=0}^{n-1} \frac{(z\overline{w})^k}{k!}.$$

As $n \to \infty$, we get a determinantal process with the kernel

$$\begin{aligned} K(z,w) &= \frac{1}{\pi} e^{-\frac{1}{2}(|z|^2+|w|^2)} \sum_{k=0}^{\infty} \frac{(z\overline{w})^k}{k!}. \\ &= \frac{1}{\pi} e^{-\frac{1}{2}|z-w|^2}. \end{aligned}$$

Since $K(z,w)$ depends only on $|z-w|$, the distribution of the process is translation invariant.

Zeros of the *i.i.d.* **Gaussian power series** [Virág-P.].
Let

$$\begin{aligned} f_U(z) &= \sum_{n=0}^{\infty} a_n z^n \\ Z_U &= \text{zeros}(f_U) \end{aligned} \tag{14}$$

with $\{a_n\}$ complex Gaussian, density$(re^{i\theta}) = e^{-r^2}$.

Theorem: (Hannay, Zelditch-Shiffman, ...)
Law of $\mathbf{Z}_U$ invariant under Möbius transformations $z \longrightarrow e^{i\alpha}\frac{z-\lambda}{1-\bar{\lambda}z}$ that preserve unit disk.

Euclidean analog:

$$f_{\mathcal{C}} = \sum_{n=0}^{\infty} \frac{a_n z^n}{\sqrt{n!}}, \tag{15}$$

satisfies

$$\begin{aligned} \mathrm{Cov}[f_{\mathcal{C}}(z), f_{\mathcal{C}}(w)] &= \mathbf{E}\left[\textstyle\sum_n \frac{a_n z^n}{\sqrt{n!}} \cdot \sum_k \frac{a_k \bar{w}^k}{\sqrt{k!}}\right] \\ &= \textstyle\sum_{n=0}^{\infty} \frac{z^n \bar{w}^n}{n!} = e^{z\bar{w}}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Cov}[f_{\mathcal{C}}(z+a), f_{\mathcal{C}}(w+a)] &= e^{(z+a)(\bar{w}+\bar{a})} \\ &= \mathrm{Cov}\left[e^{|a|^2/2} e^{\bar{a}z} f_{\mathcal{C}}(z), e^{|a|^2/2} e^{\bar{a}w} f_{\mathcal{C}}(w)\right]. \end{aligned}$$

Since Gaussian processes are determined by $\mathrm{Cov}(\cdot,\cdot)$ this proves translation invariance of $\mathrm{Law}[\mathrm{zeros}(f_{\mathcal{C}})]$.

Definition: Let $p_\epsilon(z_1, \ldots, z_n)$ denote the probability that a random function f has zeros in $B_\epsilon(z_1), \ldots B_\epsilon(z_n)$. **Joint intensity** of zeros (if it exists) is defined to be

$$p(z_1, \ldots, z_n) = \lim_{\epsilon \downarrow 0} \frac{p_\epsilon(z_1, \ldots, z_n)}{(\pi \epsilon^2)^n} \tag{16}$$

Theorem: (Hammersley)
Let f be a Gaussian analytic function in a planar domain D, $z_1, \ldots, z_n \in D$, and consider the matrix $A = \left(\mathbf{E} f(z_i)\overline{f(z_j)}\right)$. If A is non-singular then $p(z_1, \ldots z_n)$ exists and equals

$$\frac{\mathbf{E}\left(|f'(z_1)\cdots f'(z_n)|^2 \;\middle|\; f(z_1) = \cdots = f(z_n) = 0 \right)}{\det(\pi A)}.$$

Geometric proof: (Edelman, Kostlan)

We restrict to real polynomials. Set

$$p(t) = a_0\sigma_0 + a_1\sigma_1 t + \ldots + a_n\sigma_n t^n \tag{17}$$

with $a_k \sim N(0,1)$, independent. Define $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ and $\mathbf{v}(t) = (\sigma_0, \sigma_1 t, \ldots, \sigma_n t^n)$. Let $\alpha = \frac{\mathbf{a}}{\|\mathbf{a}\|}$ and $\gamma(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$. Then t is a zero of $p(t)$ iff $\alpha \cdot \gamma(t) = 0$. Define

$$\gamma_\perp(t) = \text{points on } S_{n+2} \perp \text{ to } \gamma(t).$$

Since a_k are *i.i.d.* $N(0,1)$, α is uniform on S_{n+2}. Hence, the expected number of zeros in $[a,b]$ is:

$$\int_{S_{n+2}} \#\left\{t : t \in [a,b], z \in \gamma_\perp(t)\right\} d\mu(z) \tag{18}$$

where μ is uniform on S_{n+2}.

This integral computes the area swept out by $\gamma_\perp(t)$ on $[a,b]$ counting multiplicity. If $\gamma(t)$ lies along a geodesic then

$$\frac{\text{Area swept out on } [a,b]}{\text{Area } S_{n+2}} = \frac{1}{\pi}(\text{Length of } \gamma_{[a,b]}(t)) \tag{19}$$

Approximating $\gamma(t)$ with piecewise geodesic arcs, we see that (19) holds for any rectifiable curve. Thus,

$$\mathbf{E}(\#\text{ zeros in } [a,b]) = \frac{1}{\pi}\int_a^b \| \gamma'(t) \| \, dt \tag{20}$$

so the intensity of zeros is $\frac{\| \gamma'(t) \|}{\pi}$.

* This argument generalizes to the case of complex zeros.
* More difficult to generalize computation to find higher order intensities ($n > 1$).

Linear approximation proof: We give the proof for $n = 1$, the generalization to $n > 1$ is straightforward. Set $\tilde{f}(z) = bz + a$, where $b = f'(z_1)$ and $a = f(z)$. Write $b = ca + \beta$, where β and a are independent (β is the projection of b onto the orthocomplement of the subspace spanned by a), take $D_\epsilon = \left\{|\beta| < \epsilon^{-\delta}\right\}$ and let $Z = -a/b$ be the zero of $\tilde{f}$. Now compute:

$$
\begin{aligned}
\mathbf{P}(|Z| < \epsilon | D_\epsilon, \beta) &= \mathbf{P}(a/(ca + \beta) | D_\epsilon, \beta) \\
&= \mathbf{P}(|a/\beta| < \epsilon | D_\epsilon, \beta) + O(\epsilon^{3(1-\delta)}) \\
&= \pi \epsilon^2 \beta^2 g(0) + O(\epsilon^{3(1-\delta)})
\end{aligned}
$$

where g is the density of a. $\mathbf{P}(D_\epsilon^c)$ decays exponentially in $\epsilon^{-2\delta}$, taking expectations and using $g(0) = \frac{1}{\pi |a|^2}$ gives:

$$
\mathbf{P}(|Z| < \epsilon) = \pi \epsilon^2 \frac{\mathbf{E}(|b|^2 | a = 0)}{\pi \mathbf{E}|a|^2} + O(\epsilon^{3(1-\delta)}) \tag{21}
$$

If A and $\tilde{A}$ are the events that f and $\tilde{f}$ have a zero in $\mathcal{B}_\epsilon(z_1)$ and $\mathcal{B}_\epsilon(0)$ respectively, then $\mathbf{P}(A \triangle \tilde{A}) = o(\epsilon^2)$. We conclude that

$$p(z_1) = \frac{\mathbf{E}(|f'(z_1)|^2 | f(z_1) = 0)}{\pi \mathbf{E}|f(z_1)|^2}. \tag{22}$$

Theorem: (Virág - P.)
The joint intensity of zeros for f_U is

$$\begin{aligned} p(z_1, \ldots, z_n) &= \pi^{-n} det \left[\frac{1}{(1 - z_i \bar{z_j})^2} \right]_{i,j} \\ &= det[K(z_i, z_j)] \end{aligned}$$

where $K(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}$ is the Bergman kernel for U.

Proof of Determinantal Formula

Let

$$T_\beta(z) = \frac{z-\beta}{1-\overline{\beta}z} \tag{23}$$

denote a Möbius transformation fixing the unit disk. Also, for fixed $z_1, \ldots, z_n \in \mathbf{U}$ denote

$$\Upsilon(z) = \prod_{j=1}^{n} T_{z_j}(z). \tag{24}$$

Key facts:

1. Let $f = f_U$ and $z_1, \ldots, z_n \in \mathbf{U}$. The distribution of the random function $T_{z_1}(z) \cdots T_{z_n}(z) f(z)$ is the same as the conditional distribution of $f(z)$ given $f(z_1) = \ldots = f(z_n) = 0$.

2. It follows that the conditional joint distribution of the random variables $\big(f'(z_k) : k = 1, \ldots, n\big)$ given $f(z_1) = \ldots = f(z_n) = 0$, is the same as the unconditional joint distribution of $\big(\Upsilon'(z_k) f(z_k) : k = 1, \ldots, n\big)$.

3. Consider the $n \times n$ matrices

$$\begin{aligned} A_{jk} &= \mathbf{E} f(z_j)\overline{f(z_k)} = (1 - z_j\overline{z}_k)^{-1}, \\ M_{jk} &= (1 - z_j\overline{z}_k)^{-2}. \end{aligned}$$

By the classical Cauchy determinant formula,

$$\begin{aligned} \det(A) &= \prod_{k,j} \frac{1}{1 - \overline{z}_j z_k} \prod_{k<j} (z_k - z_j)(\overline{z}_k - \overline{z}_j) \\ &= \prod_{k=1}^{n} |\Upsilon'(z_k)|. \qquad (25) \end{aligned}$$

4. We also use Borchardt's identity:

$$\operatorname{perm}\left(\frac{1}{x_j+y_k}\right)_{j,k} \det\left(\frac{1}{x_j+y_k}\right)_{j,k} = \det\left(\frac{1}{(x_j+y_k)^2}\right)_{j,k}$$

setting $x_j = z_j^{-1}$ and $y_k = -\overline{z_k}$ and dividing both sides by $\prod_j z_j^2$, gives that

$$\operatorname{perm}(A)\det(A) = \det(M). \tag{26}$$

5. Finally, recall the Gaussian moment formula: If $Z_1, \ldots, Z_n$ are jointly complex Gaussian random variables with covariance matrix $C_{jk} = \mathbf{E} Z_j \overline{Z_k}$, then $\mathbf{E}\big(|Z_1 \cdots Z_n|^2\big) = \operatorname{perm}(\mathcal{C})$.

From Hammersley's formula $p(z_1, \ldots, z_n)$ equals

$$\frac{\mathbf{E}\Big(|f'(z_1) \cdots f'(z_n)|^2 \,\Big|\, f(z_1),\ldots,f(z_n)=0 \Big)}{\pi^n \det(A)} .$$

The numerator equals

$$\mathbf{E}\big(|f(z_1) \cdots f(z_n)|^2\big) \prod_k |\Upsilon'(z_k)|^2 = \operatorname{perm}(A) \det(A)^2 ,$$

where the last equality uses the Gaussian moment formula. Thus,

$$\begin{aligned} p(z_1, \ldots, z_n) &= \pi^{-n} \operatorname{perm}(A) \det(A) \\ &= \pi^{-n} \det(M). \end{aligned}$$

Theorem 2: (Virág - P.)
Let

$$X_k \sim \begin{cases} 1 & r^{2k} \\ 0 & 1 - r^{2k} \end{cases}$$

be independent. Then $\sum_1^\infty X_k$ and $N_r = |\mathbf{Z}_U \cap B(0,r)|$ have same distribution.

Corollary: Let $h_r = 4\pi r^2/(1-r^2)$ (hyperbolic area). Then

$$\mathbf{P}(N_r = 0) = e^{-h_r \frac{\pi}{24} + o(h_r)} = e^{\frac{-\pi^2/12 + o(1)}{1-r}}. \tag{27}$$

All of the above generalize to other simply connected domains with smooth boundary.

$$\mathbf{E}\left(f_D(z) f_D(w)\right) = S_D(z,w) \ \text{(Szëgo Kernel)} \tag{28}$$

Dynamics

Let

$$f_{t,U} = \sum_n a_n(t) z^n \tag{29}$$

with $a_n(t)$ performing Ornstein-Uhlenbeck diffusion, $a_n(t) = e^{-t/2} W_n(e^t)$. Suppose that $\mathbf{Z}_U(0)$ contains the origin. Movement of this zero satisfies stochastic differential equation

$$dz = \sigma dW \tag{30}$$

where

$$\frac{1}{\sigma} = |f_U'(0)| = c \lim_{r \uparrow 1} \frac{1}{\sqrt{1 - r^2}} \prod_{\substack{z \in Z_U \\ 0 < |z| < r}} |z| = \tilde{c} \prod_{k=1}^{\infty} e^{1/k} |z_k|. \tag{31}$$

Denote $q = r^2$. Key to law of $N_r = |\mathbf{Z}_U \cap B(0,r)|$:

$$\begin{aligned}
\mathbf{E}\binom{N_r}{k} &= \frac{1}{k!}\int_{B_r^k} p(z_1,\ldots,z_k)dz_1,\ldots dz_k \\
&= \frac{q^{\binom{k+1}{2}}}{(1-q)(1-q^2)\ldots(1-q^k)} \\
&= \gamma_k.
\end{aligned}$$

Euler's partition identity

$$\sum_{k=0}^{\infty} \gamma_k s^k = \prod (1+q^k s), \tag{32}$$

implies that

$$\mathbf{E}(1+s)^{N_r} = \sum_{k=0}^{\infty} \mathbf{E}\binom{N_r}{k} s^k = \sum \gamma_k s^k \tag{33}$$

has product form!

Theorem:(Virág-P. (2003)) The set $\{|z_1|, |z_2|, |z_3|, \ldots\}$ of moduli of the zeroes of the GAF $\sum_{n=0}^{\infty} a_n z^n$ has the same distribution as $\{U_1^{1/2}, U_2^{1/4}, U_3^{1/6}, \ldots\}$ where U_i are i.i.d. uniform on [0,1].

Proof: (modeled after Kostlan(1992)) The zero set is determinantal with the Bergman kernel

$$K(z, w) = \frac{1}{\pi}(1 + 2(z\overline{w}) + 3(z\overline{w})^2 + 4(z\overline{w})^3 + \ldots).$$

Let K_n be the above series truncated to n terms. The determinantal processes with kernels K_n are stochastically increasing. Also the expected number of points in a region S is $\int_S K_n(x, x)dx$ which converges to $\int_S K(x, x)dx$. Hence, the point processes converge in distribution.

Conclusion: Enough to prove the analogous theorem for the process for the process with kernel

$$K_n(z,w) = \frac{1}{\pi}(1 + 2(z\overline{w}) + \ldots + n(z\overline{w})^{n-1}).$$

For any $z_1, \ldots, z_n$,

$$\begin{pmatrix} K_n(z_1,z_1) & \ldots & \ldots & K_n(z_1,z_n) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ K_n(z_n,z_1) & \ldots & \ldots & K_n(z_n,z_n) \end{pmatrix} =$$

$$\frac{n!}{\pi^n} \begin{pmatrix} 1 & z_1 & \ldots & z_1^{n-1} \\ 1 & z_2 & \ldots & z_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & z_n & \ldots & z_n^{n-1} \end{pmatrix} \begin{pmatrix} 1 & \ldots & 1 \\ \overline{z}_1 & \ldots & \overline{z}_n \\ \ldots & \ldots & \ldots \\ \overline{z}_1^{n-1} & \ldots & \overline{z}_n^{n-1} \end{pmatrix}.$$

Setting $z_j = r_j e^{i\theta_j}$ we get for the n-point correlation of $\{|z_j|\}_{j=1}^n$,

$$\int_{[0,2\pi]^n} \det\left((K_n(z_j, z_k))\right) \prod_j r_j d\theta_1 \ldots d\theta_n =$$

$$\frac{n!}{\pi^n} \int \left(\sum_\sigma \operatorname{sgn}(\sigma) \prod_{j=1}^n z_j^{\sigma_j - 1} \right) \left(\sum_\tau \operatorname{sgn}(\tau) \prod_{j=1}^n \overline{z}_j^{\tau_j - 1} \right) \prod_j r_j d\theta_1 \ldots d\theta_n.$$

When we expand the sums, if we take $\sigma \neq \tau$, then the integrand contains a factor of the form $z_j^p \overline{z}_j^q$ with $p \neq q$ and thus the integral vanishes. When $\sigma = \tau$, we get

$$(*) \qquad (2\pi)^{2n} \prod_j r_j^{2\sigma_j - 1}.$$

This gives us $2^n n! \sum_\sigma \prod_{j=1}^n r_j^{2\sigma_j - 1}$. Now $U_j^{\frac{1}{2j}}$ has density $2jx^{2j-1}$ in $[0,1]$. Hence the n-point correlation of $\{U_1^{1/2}, \ldots, U_n^{1/2n}\}$ is precisely $(*)$.

Reconstruction of $|f_{\mathbf{U},\rho}|$ from its zeros

Theorem

(i) Let $\rho > 0$. Consider the random function $f_{\mathbf{U},\rho}$, and order its zero set $Z_{\mathbf{U},\rho}$ in increasing absolute value, as $\{z_k\}_{k=1}^{\infty}$. Then

$$|f_{\mathbf{U},\rho}(0)| = c_\rho \prod_{k=1}^{\infty} e^{\rho/2k} |z_k| \quad a.s. \tag{34}$$

where $c_\rho = e^{(\rho-\gamma-\gamma\rho)/2}\rho^{-\rho/2}$ and $\gamma = \lim_n \Big(\sum_{k=1}^{n} \frac{1}{k} - \log n \Big)$ is Euler's constant.

(ii) More generally, Given $\zeta \in \mathbf{U}$, let $\{\zeta_k\}_{k=1}^{\infty}$ be $Z_{\mathbf{U},\rho}$, ordered in increasing hyperbolic distance from ζ. Then

$$|f_{\mathbf{U},\rho}(\zeta)| = c_\rho (1 - |\zeta|^2)^{-\rho/2} \prod_{k=1}^{\infty} e^{\rho/2k} \left| \frac{\zeta_k - \zeta}{1 - \bar{\zeta}\zeta_k} \right|. \tag{35}$$

The analytic extension of white noise.

Claim: Up to the constant term, the power series $f_{\mathbf{U}}$ has the same distribution as the analytic extension of white noise on the unit circle.

Proof: Let $B(\cdot)$ be a standard real Brownian motion, and let

$$u(z) = \int_0^{2\pi} \mathrm{Poi}(z, e^{it}) dB(t),$$

where

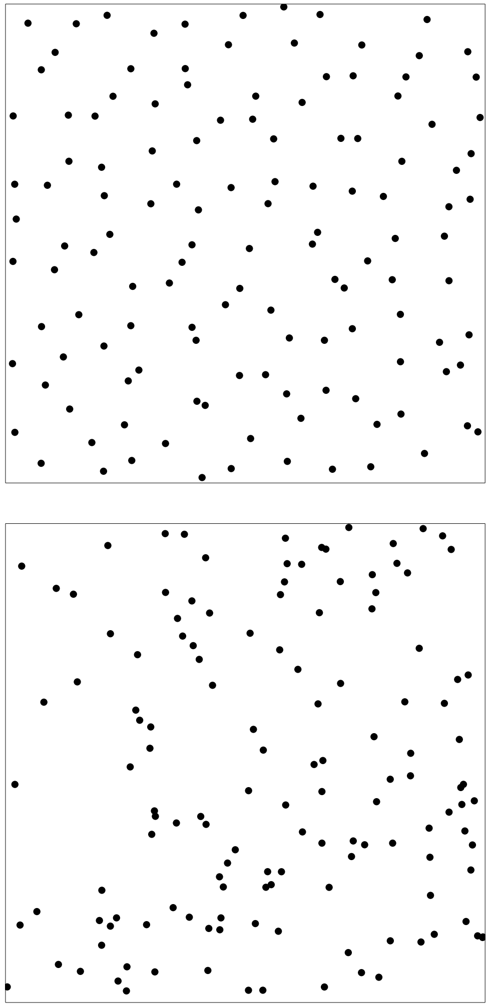
$$\mathrm{Poi}(z, w) = \frac{1}{2\pi} \Re \left(\frac{1 + z\overline{w}}{1 - z\overline{w}} \right)$$

is the Poisson kernel.

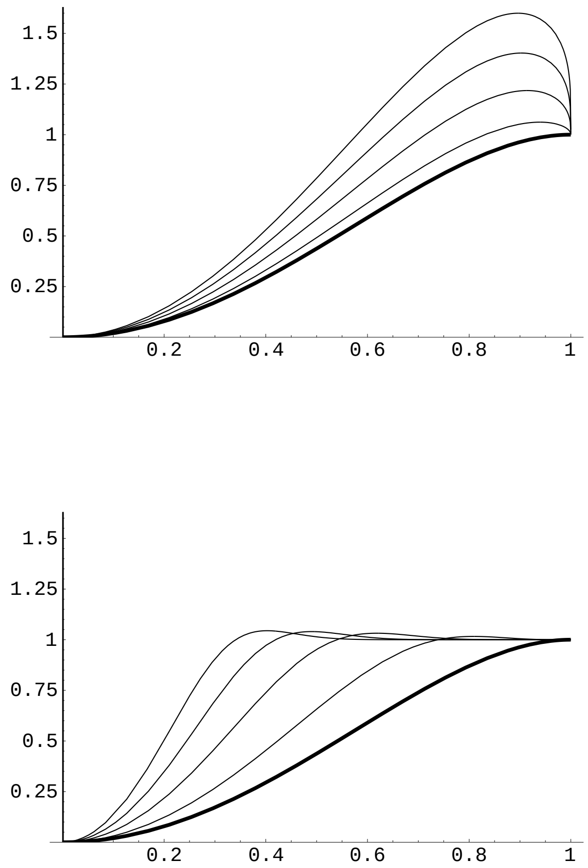
Hence, $\mathbf{E}\left(u(z)u(w)\right) = \text{Poi}(z,w)$, by the kernel property of the Poi. Therefore if b is a standard real Gaussian independent of $B(\cdot)$, then

$$\tilde{u}(z) = \sqrt{\frac{\pi}{2}}u(z) + \frac{b}{2} \tag{36}$$

has covariance structure $\mathbf{E}[\tilde{u}(z)\tilde{u}(w)] = \frac{1}{2}\Re\left(\frac{1}{1-z\overline{w}}\right)$. Thus $\tilde{u}$ has the same distribution as $\Re f_{\mathbf{U}}$.



The translation invariant root process and a Poisson point process with the same intensity on the plane



Relative intensity at $z = 0$ and $z = r$ as a function of r for $\rho = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ and for $\rho = 1, 4, 9, 16, 25$.