Optimal mean width and metric entropy estimates for convex bodies

(joint work w/ Grigoris Paouris)

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The “size” of convex bodies

convex bodies: $K \subset \mathbf{R}^n$ is a *convex body* if it is compact, convex, and has nonempty interior

MAIN QUESTION OF THIS TALK

how “large” may a convex body $K \subset \mathbf{R}^n$ be, when its volume is fixed?

two notions of size:

1. *covering number* — how many ellipsoids of similar volume are required to cover K?
2. *volume of projections* — what is the average size of a k-dimensional projection of K?

(we will see that these two notions are, of course, very closely related)

Outline

1. Covering a convex body

2. Projections of a convex body

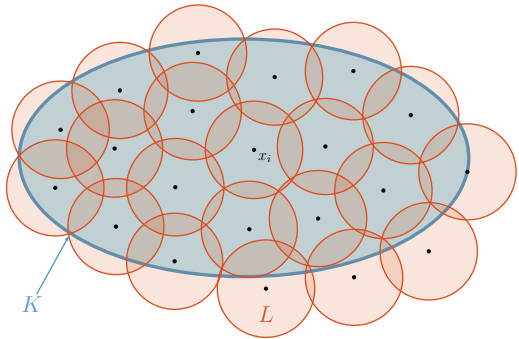
3. Proof ideas

1. Covering a convex body

Covering number and metric entropy

covering number: for compact sets $K, L \subset \mathbf{R}^n$ with L having nonempty interior,

$$N(K, L) = \min \left\{ m \geqslant 1 : \exists x_1, \ldots, x_m \in \mathbf{R}^n \text{ such that } K \subset \bigcup_{i=1}^{m} x_i + L \right\}$$



metric entropy: logarithm of covering number, $\log N(K, L)$

Background on metric entropy and covering numbers

covering numbers:

- typically attributed to [Kolmogorov, '56; Kolmogorov-Tihomirov, '59], who wanted to understand how to succinctly approximate a function to a given level of accuracy
- appeared in metric space setting in [Pontrjagin-Schnirelmann, '32; von Neumann, '42]

entropy numbers and compactness of operators:

- given linear operator $u\colon X \to Y$ between Banach spaces, involves the study of

$$r \mapsto N(u(B_X), rB_Y), \quad r > 0,$$

  where B_X, B_Y are unit balls of spaces X, Y
- leads to the *entropy numbers*, defined as

$$e_k(u) = \inf\{r > 0 : N(u(B_X), rB_Y) \leqslant 2^{k-1}\}, \quad k \geqslant 1$$

- quantifies compactness: note that u is compact iff $e_k \downarrow 0$ as $k \to \infty$
- typically attributed to [Pietsch, '72] though also appeared in [Mitjagin, '61; Mitjagin-Pelczynski, '68; Triebel, '70]

Background on metric entropy and covering numbers

existence of M-ellipsoids:

- [Milman, '86] established for convex bodies $K \subset \mathbf{R}^n$ which are centrally symmetric (*i.e.*, $K = -K$) that there exists an ellipsoid $\mathcal{E}$ such that:

$$|\mathcal{E}| = |K| \quad \text{and} \quad N(K, \mathcal{E}) \leqslant \exp(Cn).$$

- [Milman-Pajor, 2000] obtained the same result in the non-symmetric case, up to translations

regular M-ellipsoids:

- [Pisier, '89a; Pisier, '89b] showed the existence of *regular* M-ellipsoids: for a centrally symmetric convex body $K \subset \mathbf{R}^n$ and each $\alpha \in (0, 2)$, there exists an ellipsoid $\mathcal{E}_\alpha$ such that

$$\max\{N(K, r\mathcal{E}_\alpha), N(\mathcal{E}_\alpha, rK), N(K^\circ, r\mathcal{E}_\alpha^\circ), N(\mathcal{E}_\alpha^\circ, rK^\circ)\} \leqslant \exp(C_\alpha n / r^\alpha), \quad \text{for } r \geqslant C_\alpha^{1/\alpha}$$

- weaker generalizations for non-symmetric bodies appeared in [Vritsiou, 2024]

duality of entropy:

- Pietsch's duality of entropy conjecture was partially resolved [Artstein-Milman-Szarek, 2004]:

$$N(\mathcal{E}^\circ, \tfrac{1}{c}K^\circ)^{1/C} \leqslant N(K, \mathcal{E}) \leqslant N(\mathcal{E}^\circ, cK^\circ)^C$$

  for any centrally symmetric convex body $K \subset \mathbf{R}^n$, any ellipsoid $\mathcal{E}$

Covering numbers of convex bodies

THEOREM [PAOURIS-P., 2026]

for any convex body $K_0 \subset \mathbf{R}^n$, there is a translate K and ellipsoid $\mathcal{E}$ of the same volume such that

$$\max\{N(K, r\mathcal{E}), N(K^\circ, r\mathcal{E}^\circ), N(\mathcal{E}, rK), N(\mathcal{E}^\circ, rK^\circ)\}$$
$$\leqslant \inf_{|\tilde{\mathcal{E}}|=|B_1^n|} \max\{N(B_1^n, cr\tilde{\mathcal{E}}), N(B_\infty^n, cr\tilde{\mathcal{E}}^\circ), N(\tilde{\mathcal{E}}, crB_1^n), N(\tilde{\mathcal{E}}^\circ, crB_\infty^n)\}^C,$$

holds for every $r > 0$

some remarks:

- infimum over ellipsoids $\tilde{\mathcal{E}}$ is attained (up to constants) with suitably scaled ball B_2^n
- the ellipsoid $\mathcal{E}$ does *not* depend on r
- result does *not* require symmetry of body K
- may replace crosspolytope B_1^n with any regular n-simplex S_n (after translation)

Equivalent formulation via entropy numbers

entropy numbers:

- *entropy numbers of a convex body* $K \subset \mathbf{R}^n$ are

$$e_k(K) = \inf\{r > 0 : N(K, rB_2^n) \leqslant 2^{k-1}\}, \qquad e_k^*(K) = \inf\{r > 0 : N(B_2^n, rK^\circ) \leqslant 2^{k-1}\}$$

- for centrally symmetric K, these correspond to $e_k(\mathrm{id})$, $e_k(\mathrm{id}^*)$ for $\mathrm{id}\colon (\mathbf{R}^n, \|\cdot\|_K) \to (\mathbf{R}^n, \|\cdot\|_2)$
- we set $E_k(K) = \max\{e_k(K), e_k^*(K), e_k(K^\circ), e_k^*(K^\circ)\}$

equivalent re-formulation of previous theorem:

THEOREM [PAOURIS-P., 2026]

for any convex body $K \subset \mathbf{R}^n$, there is an affine map ψ such that $|\psi(K)| = |B_2^n|$, and

$$E_k(\psi(K)) \lesssim \inf_{|\tilde{\psi}(B_1^n)| = |B_2^n|} E_k(\tilde{\psi}(B_1^n)), \quad \text{for all } k \geqslant 1$$

Comparison with the regular positions of Pisier

Pisier's extension of V. Milman's M-ellipsoid theorem [Pisier, '89a] can be formulated as follows

THEOREM [PISIER, '89A]

for any centrally symmetric convex body $K \subset \mathbf{R}^n$, any $\beta > 1/2$,
there exists $T_\beta \in \mathrm{GL}(n)$ such that $|T_\beta K| = |B_2^n|$ and:

$$E_k(T_\beta K) \lesssim \max\left\{1, \frac{1}{\sqrt{2\beta - 1}}\right\}\left(\frac{n}{k}\right)^\beta, \quad \text{for } k \in \{1, 2, \ldots, n\}$$

some remarks:

- fixing $k \leqslant n$ and then optimizing in β, Pisier's theorem implies

$$\inf_{|S_k K| = |B_2^n|} E_k(S_k K) \lesssim \sqrt{\frac{n}{k} \log\left(\frac{en}{k}\right)}$$

- in fact, essentially by [Schütt, '84], our theorem implies existence of $T \in \mathrm{GL}(n)$

$$|TK| = |B_2^n|, \quad E_k(TK) \lesssim \min\left\{\sqrt{n}, \sqrt{\frac{n}{k} \log\left(\frac{en}{k}\right)}\right\} \quad \textit{simultaneously for } k \in \{1, 2, \ldots, n\}$$

- our theorem also implies the above in non-symmetric case, after a translation

2. Projections of a convex body

Projections of a convex body: mean width and quermassintegrals

QUERMASSINTEGRALS OF A CONVEX BODY

$$W_{[k]}(K) = \mathbf{E}_{F\sim\nu_{n,k}}\left[\frac{|P_F K|}{|B_2^k|}\right]^{1/k}, \qquad \text{for } k \in \{1,2,\dots,n\}$$

- $\nu_{n,k}$ denotes uniform (Haar) probability measure on all k-dimensional subspaces of $\mathbf{R}^n$
- P_F denotes orthogonal projection onto k-dimensional subspace $F \subset \mathbf{R}^n$

important special cases:

- for $k = 1$, this is the *spherical mean width*

$$W_{[1]}(K) = \mathbf{E}_{\theta\sim\sigma_{n-1}}\left[\sup_{x\in K}\langle x,\theta\rangle\right] \equiv M^*(K)$$

- for $k = n - 1$, this is the (scaled) *surface area*,

$$W_{[n-1]}(K) = \left(\frac{|\partial K|}{|\partial B_2^n|}\right)^{1/(n-1)}$$

- for $k = n$, this is the *volume radius*,

$$W_{[n]}(K) = \left(\frac{|K|}{|B_2^n|}\right)^{1/n} \equiv \mathrm{vr}(K)$$

Background on mean width and quermassintegrals

- *Steiner formula*: for a convex body $K \subset \mathbf{R}^n$ and any $t \geqslant 0$,

$$|K + tB_2^n| = \sum_{k=0}^{n} |B_2^k| V_{n-k}(K) t^k$$

- coefficients $\{V_k(K)\}_{k=0}^n$ are referred to as *intrinsic volumes* and satisfy

$$V_k(K) = c_{n,k} \cdot W_{[k]}(K)^k \quad \text{where} \quad c_{n,k} = \binom{n}{k} \frac{|B_2^n|}{|B_2^{n-k}|}$$

- our normalization and ordering of the quermassintegrals are not entirely standard
- by Alexandrov inequalities, have

$$M^*(K) \equiv \quad W_{[1]}(K) \geqslant W_{[2]}(K) \geqslant \cdots \geqslant W_{[n]}(K) \quad \equiv \mathrm{vr}(K)$$

- the inequality $M^*(K) \geqslant \mathrm{vr}(K)$ is referred to as *Urysohn's inequality*
- extremizing $W_{[k]}(K)$ over bodies of fixed volume $\longleftrightarrow$ "generalized" isoperimetric problems

Background on mean width and quermassintegrals

- for $k = n - 1$, the classical isoperimetric theorem implies that

$$W_{[n-1]}(K) \geqslant \mathrm{vr}(K),$$

  with equality iff K is the ball $\mathrm{vr}(K)B_2^n$ (up to translation)
- in the other direction, a theorem of [Ball, '91] shows

$$\inf_{T \in \mathrm{SL}(n)} W_{[n-1]}(TK) \leqslant W_{[n-1]}(S_n), \quad \text{where } |K| = |S_n|,$$

  and S_n is a regular n-simplex; equality holds iff K is a simplex [Barthe, '98]
- maximal order of the minimal kth quermassintegral, *i.e.*,

$$A_{n,k} = \sup_K \inf_{T \in \mathrm{SL}(n)} \frac{W_{[k]}(TK)}{\mathrm{vr}(K)}$$

  was (previously) unknown for $k \in \{1, 2, \ldots, cn\}$
- $k \gtrsim n$: can show $A_{n,k} \simeq 1$, via the work of [Milman, '86; Milman-Pajor, 2000]
- our methods, unfortunately, are unlikely to yield exact constants, equality cases

Volumes of typical k-dimensional projections of convex bodies

THEOREM [PAOURIS-P., 2026]

for any convex body $K_0 \subset \mathbf{R}^n$, there is a translate K and $T \in \mathrm{SL}(n)$ such that

$$\max\left\{\frac{W_{[k]}(TK)}{\mathrm{vr}(K)}, \frac{W_{[k]}((TK)^\circ)}{\mathrm{vr}(K^\circ)}\right\} \lesssim \inf_{\tilde{T}\in\mathrm{SL}(n)} \max\left\{\frac{W_{[k]}(\tilde{T}B_1^n)}{\mathrm{vr}(B_1^n)}, \frac{W_{[k]}((\tilde{T}B_1^n)^\circ)}{\mathrm{vr}(B_\infty^n)}\right\},$$

simultaneously, for every $k \in \{1, 2, \dots, n\}$

some remarks:

- essentially by a computation in [Paouris-Pivovarov, 2013], it holds that

$$\inf_{\tilde{T}\in\mathrm{SL}(n)} \max\left\{\frac{W_{[k]}(\tilde{T}B_1^n)}{\mathrm{vr}(B_1^n)}, \frac{W_{[k]}((\tilde{T}B_1^n)^\circ)}{\mathrm{vr}(B_\infty^n)}\right\} \simeq \sqrt{\log\left(\frac{en}{k}\right)}$$

- important special case is *minimal mean width*: for any convex body $K \subset \mathbf{R}^n$,

$$1 \leqslant \inf_{T\in\mathrm{SL}(n)} \frac{M^*(TK)}{\mathrm{vr}(K)} \lesssim \sqrt{\log(en)},$$

- in all the results above, may replace B_1^n by a regular n-simplex, up to translation

History of minimal mean width

- the question about the order of the minimal mean width of convex bodies seems to have first appeared in [Giannopoulos-Milman-Rudelson, 2000]
- it is mentioned in the books [Aubrun-Szarek, 2017; Artstein-Avidan-Giannopoulos-Milman, 2015]
- following the introduction of the K-convexity $\mathbf{K}(K)$ constant, [Figiel-Tomczak-Jaegermann, '79] showed that in the centrally symmetric case,

$$\inf_{T\in \mathrm{GL}(n)} M^*(TK)M(TK) \lesssim \mathbf{K}(K) \lesssim \log(1+d_{\mathrm{BM}}(K,B_2^n)) \lesssim \log(en)$$

- in the centrally symmetric case, as $M(TK) \gtrsim \mathrm{vr}(TK)^{-1}$, it implies

$$\inf_{T\in \mathrm{GL}(n)} \frac{M^*(TK)}{\mathrm{vr}(K)} \lesssim \log(en),$$

  for the non-symmetric case, also follows via Rogers-Shephard inequality

- [Bizeul, 2026] established maximal order of $M^*(K)/\mathrm{vr}(K)$ among bodies K in isotropic position; also implies order of minimal mean width for general convex bodies

3. Proof ideas

Convex comparison of continuous martingales

the following result seems to date (at least) back to [Borkar, '87]

PROPOSITION

for any $\mathbf{R}^n$-valued, continuous L^2 martingale $\{M_t\}_{0 \leqslant t \leqslant T}$ with $M_0 = 0$,
if $\mathrm{d}[M]_t \succeq \rho I_n \mathrm{d}t$ for some $\rho > 0$, then, for every convex $F \colon \mathbf{R}^n \to \mathbf{R}$,

$$\mathbf{E}\, F(M_T) \geqslant \mathbf{E}_{G \sim \gamma_n} F\Big(\sqrt{\rho T}\, G\Big)$$

some remarks:

- we apply with $F = h_K$, for some convex body K containing the origin in its interior; in that case,

$$\sqrt{n} M^*(K) \lesssim \frac{1}{\sqrt{\rho T}} \cdot \mathbf{E}\, h_K(M_T) = \frac{1}{\sqrt{\rho T}} \cdot \mathbf{E}\, \|M_T\|_{K^\circ}$$

- we will arrange that the body and martingale $\{M_t\}_{t \geqslant 0}$ satisfy

$$|K^\circ|^{1/n} \simeq 1, \quad \mathbf{E}\, \|M_T\|_{K^\circ} \lesssim 1 \quad \text{for} \quad T \simeq \frac{1}{\log(en)}$$

and that the proposition holds with $\rho \gtrsim 1$

The stochastic localization process

our martingales are constructed via stochastic localization process [Eldan, 2012; Eldan, 2013; Lee-Vempala, 2024]

CONSTRUCTION OF STOCHASTIC LOCALIZATION PROCESS

given a centered probability measure μ on $\mathbf{R}^n$, study tilt measures

$$\mathrm{d}\mu_{t,\eta} \propto \mathrm{e}^{\langle x,\eta\rangle - t\|x\|_2^2/2}\,\mathrm{d}\mu, \quad a_t(\eta) = \mathbf{E}_{X\sim\mu_{t,\eta}}[X], \quad A_t(\eta) = \mathrm{Cov}_{X\sim\mu_{t,\eta}}(X)$$

construct the *random tilt process*,

$$\mathrm{d}\theta_t = a_t(\theta_t)\mathrm{d}t + \mathrm{d}B_t, \quad \theta_0 = 0$$

leads to random measure, mean, covariance process:

$$\mu_t = \mu_{t,\theta_t}, \quad a_t = a_t(\theta_t), \quad A_t = A_t(\theta_t)$$

Interpretation of the stochastic localization process

filtering interpretation: (observed by [El Alaoui-Montanari, 2022; Klartag-Putterman, 2023])

- it holds that
$$\mathrm{Law}\Big(\{tX + B_t\}_{t\geqslant 0}\Big) = \mathrm{Law}\Big(\{\theta_t\}_{t\geqslant 0}\Big),$$
where $X \sim \mu$ independently of $\{B_t\}_{t\geqslant 0}$
- also, $\mu_t = \mathrm{Law}(X \mid tX + B_t) = \mathrm{Law}(X \mid X + t^{-1}B_t)$ for $t > 0$
- pathwise: essentially "interpolate" between μ (at $t \to 0^+$) and δ_X (as $t \to \infty$)

two key processes: for μ supported on (sufficiently smooth) convex body L,

- *mean process*:
$$a_t = \int_L x\, p_t(x)\,\mathrm{d}x = \int_0^t A_s\,\mathrm{d}B_s, \quad \mathrm{d}[a]_t = A_t^2\,\mathrm{d}t$$
- *normal process*:
$$a_t^\circ = \int_{\partial L} n_L(x) p_t(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_L \nabla \log p_t(x)\, p_t(x)\,\mathrm{d}x$$

Bobkov's maximal Gaussian measure position

[Bobkov, 2011] defined a c.s. convex body $K \subset \mathbf{R}^n$ to be in *maximal Gaussian measure* position if

$$\gamma_n(K) \geqslant \gamma_n(TK) \quad \text{for all } T \in \mathrm{SL}(n)$$

GENERALIZATION TO NON-SYMMETRIC CASE ('B-POSITION')

a (possibly non-symmetric) convex body $K \subset \mathbf{R}^n$ is in *maximal Gaussian measure position* if

$$\gamma_n(K) \geqslant \gamma_n(TK + x) \quad \text{for all } (T, x) \in \mathrm{SL}(n) \times \mathbf{R}^n$$

properties:

- equivalent characterization via isotropy: if γ_K denotes γ_n conditioned to K

$$\mathbf{E}_{X \sim \gamma_K}[X] = 0 \quad \text{and} \quad \mathbf{E}_{X \sim \gamma_K}[X \otimes X] = \alpha_K^2 I_n$$

- if $|K|^{1/n} \simeq 1$, then B-position is Milman's position
- for a convex body $K \subset \mathbf{R}^n$ there always exists a volume-preserving affine image of K which is in B-position; unique up to orthogonal transformations

A non-symmetric variant of the strong Gaussian (B)-theorem

the properties on the last slide can be established from the following result

PROPOSITION [PAOURIS-P., 2026]

for any real, symmetric $A \in \mathbf{R}^{n\times n}$ and any convex body $K \subset \mathbf{R}^n$, the map

$$t \mapsto \sup_{x\in\mathbf{R}^n} \gamma_n(\mathrm{e}^{tA}K + x)$$

is log-concave on $\mathbf{R}$

some remarks:

- the result above is false without optimizing over translates [Nayar-Tkocz, 2013]
- proof of the result:
  - compute second derivative of the logarithm, realize it is nonpositive iff

$$\mathrm{Var}_{\gamma_K}(f) \leqslant \frac{1}{2}\Big(\mathbf{E}_{\gamma_K}\|\nabla f\|_2^2 + \|\mathbf{E}_{\gamma_K}\nabla f\|_2^2\Big), \quad \text{for certain } f;$$

  - follows by adapting proof of [Cordero-Erausquin-Fradelizi-Maurey, 2004; Barthe-Cordero-Erausquin, 2013]

Comparison theorem and width of K

THEOREM

for any convex body K with K° in B-position with $|K^\circ| = 1$,

$$\mathbf{E}_{\gamma_n} F(G) \lesssim \inf_{u \in (0,n]} \Big(\sqrt{\log(en/u)}\, \mathbf{E}_{X \sim \gamma_{K^\circ}} F(X) + \mathrm{Lip}(F)\sqrt{u} \Big),$$

for any positively 1-homogeneous, convex $F \colon \mathbf{R}^n \to \mathbf{R}_+$

comments:

- we replace $\mathrm{d}a_t = A_t \mathrm{d}B_t$ by $\mathrm{d}\tilde{a}_t = \tilde{A}_t \mathrm{d}B_t$ with $\tilde{A}_t$ explicitly truncated from below
- the first term: arises from the convex comparison for continuous martingales
- the error term $\mathrm{Lip}(F)\sqrt{u}$: by time $T \simeq (\log(en/u))^{-1}$, have $\approx u$ truncated eigenvalues
- implies that

$$\sup_{x \in K} M^*\big(K \cap (x + r\,\mathrm{vr}(K) B_2^n)\big) \lesssim \mathrm{vr}(K)\sqrt{\log(\mathrm{e}r^2)} \qquad (\forall r \geqslant 1)$$

Comparison theorem and width of K°

THEOREM

for any convex body K with K° in B-position with $|K^\circ| = 1$,

$$\mathbf{E}_{\gamma_n} F(G) \lesssim \inf_{u\in(0,n]} \Big(n\sqrt{\log(en/u)}\, \mathbf{E}_{\nu_{K^\circ}} F(\nabla h_K(X)) + \mathrm{Lip}(F)\sqrt{u} \Big),$$

for any positively 1-homogeneous, convex $F\colon \mathbf{R}^n \to \mathbf{R}_+$.
(Here ν_{K° is the probability measure on ∂K° with $\mathrm{d}\nu_{K^\circ}(x) \propto \langle x, n_{K^\circ}(x)\rangle \mathrm{e}^{-\|x\|_2^2/2}\, \mathrm{d}\mathcal{H}^{n-1}(x)$)

comments:

- the normal process satisfies $\mathrm{d}a_t^\circ = [I_n - (1+t)A_t]\mathrm{d}B_t$; we now truncate eigenvalues of $(1+t)A_t$ from above
- the first term: arises from convex comparison, factor n comes from isotropy and volume normalization
- the error term $\mathrm{Lip}(F)\sqrt{u}$: up to $T \simeq (\log(en/u))^{-1}$, have $\approx u$ truncated eigenvalues
- implies that

$$\sup_{x\in K^\circ} M^*\left(K^\circ \cap (x + r\,\mathrm{vr}(K^\circ)B_2^n)\right) \lesssim \mathrm{vr}(K^\circ)\sqrt{\log(\mathrm{e}r^2)} \quad (\forall r \geqslant 1)$$

Reduction to the truncation bodies

truncation bodies: for a convex body $L \subset \mathbf{R}^n$, we refer to

$$L \cap (x + rB_2^n), \quad \text{for } x \in L, r > 0$$

as the *truncation bodies* of L

PROPOSITION [PAOURIS-P., 2026]

suppose that $L \subset \mathbf{R}^n$ is a convex body with the origin in its interior, and with

$$\operatorname{diam}(L) \lesssim \sqrt{n}\,\operatorname{vr}(L), \quad \text{and} \quad \sup_{x \in L} M^*\left(L \cap (x + r\,\operatorname{vr}(L)B_2^n)\right) \lesssim \operatorname{vr}(L)\sqrt{\log(\mathrm{e}r^2)} \quad (\forall r \geqslant 1)$$

then for every $t > 0$,

$$\max\left\{N(L, t\operatorname{vr}(L)B_2^n), N(B_2^n, t\operatorname{vr}(L)L^\circ)\right\} \leqslant \inf_{T \in \mathrm{SL}(n)} N(TB_1^n, c\tfrac{t}{\sqrt{n}}B_2^n)^C$$

moreover, for every $k \in \{1, 2, \ldots, n\}$

$$W_{[k]}(L) \lesssim \sqrt{n}\,\operatorname{vr}(L) \inf_{T \in \mathrm{SL}(n)} W_{[k]}(TB_1^n)$$

Ideas behind reduction

bounding the metric entropy:

- by applying the Sudakov minoration to "local" packing generated by a $t/2$-packing and t-cover,

$$\log \frac{N(L, \frac{t}{2}B_2^n)}{N(L, tB_2^n)} \lesssim \frac{n}{t^2} \sup_{x \in L} M^*(L \cap (x + tB_2^n))^2$$

- by invoking the dual Sudakov minoration, we obtain

$$\log \frac{N(B_2^n, \frac{t}{2}L^\circ)}{N(B_2^n, tL^\circ)} \lesssim \frac{n}{t^2} \sup_{x \in L} M^*(L \cap (x + tB_2^n))^2$$

bounding the quermassintegrals:

THEOREM [MOURTADA, 2025]

for any convex body $K \subset \mathbf{R}^n$,

$$\log \Big(\sum_{j=0}^{n} V_j(K) \Big) \simeq \inf_{r>0} \Big\{ \sqrt{n} \sup_{x \in K} M^*(K \cap (x + rB_2^n)) + \log N(K, rB_2^n) \Big\}$$

- we apply this result to dilates λK; optimize $\lambda > 0$ so that $V_k(\lambda_k K)$ dominates the sum

Summary

1. put K° in B-*position* such that $|K^\circ| = 1$
   - exists in non-symmetric case (by variant of strong (B) theorem)
   - equivalent to centered & scalar covariance for γ_{K°
   - implies that
   $$\int_{K^\circ} x \, \mathrm{d}\gamma_{K^\circ}(x) = 0, \quad \text{and} \quad \int_{\partial K^\circ} n_{K^\circ}(x)\phi(x) \, \mathrm{d}\mathcal{H}^{n-1}(x) = 0$$
2. consider the continuous martingales
   $$a_t = \mathbf{E}_{\mu_t} X, \quad \text{and} \quad a_t^\circ = \int_{\partial K^\circ} n_{K^\circ}(x) p_t(x) \, \mathrm{d}\mathcal{H}^{n-1}(x)$$
3. by Itô computation,
   $$\mathrm{d}a_t = A_t \, \mathrm{d}B_t, \quad \text{and} \quad \mathrm{d}a_t^\circ = \left[I - (1+t)A_t\right] \mathrm{d}B_t,$$
   to apply martingale theorem requires controlling eigenvalues of covariance A_t, above and below
4. after controlling eigenvalues, apply *convex comparison theorem for continuous martingales*
5. take convex map to be *support function for truncated bodies* to get estimate for mean width
6. *reduce from mean width of truncated bodies* to metric entropy/quermassintegrals

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