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# Black-Litterman Model

This paper introduces the Black-Litterman model and its applications.

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The Black-Litterman model is an asset allocation model developed in 1990 by Fischer Black and Robert Litterman at Goldman Sachs. This model combines ideas from the Capital Asset Pricing Model (CAPM) and the Markowitz's mean-variance optimization model to provide a tool for investors to calculate the optimal portfolio weights under specified parameters.

Prior to the Black-Litterman model, investors used to input expected returns of the assets into the Markowitz's model to generate portfolio weights. However, the complex model often returned weights that did not make sense to general investors, especially when the investors had opinions about how the market or certain assets will perform in the future. The Black-Litterman model is then developed to provide neutral weights for the investors that can be adjusted according to their opinions about the market.

Before using the model, one should understand the two key assumptions behind the model. First, the model assumes that all asset returns follow the same probability distribution (usually normal distribution is selected, but investors can choose any distribution that seems fit). Second, variance of the prior and the conditional distribution about the true means of the assets and investor views are unknown.

To begin utilizing the model, investors first obtain implied market returns derived from the CAPM model. If the investors agree with the implied return, then they can use the neutral weights given by the Black-Litterman model to develop their optimal portfolio. However, if the investors disagree with the implied market returns given by CAPM, then they should utilize the Black-Litterman model, which adjusts the neutral weights according to investors' views. There are two types of market views – absolute or relative. In absolute views, investors specify the percentage return that they believe a certain asset will provide (ie. Asset 1 will have a x% return). In relative views, investors compare one asset to another in term (ie. Asset 1 will yield higher turn than asset 2 by x%). Within each view, investors also need to specify a confidence level showing how certain they feel about their views.

The Black-Litterman model expresses the investors' views and market equilibrium in terms of probability distributions. It uses the Bayesian approach to develop a probability distribution for the expected return by using CAPM equilibrium distribution as a starting point and then combining investors' views into the distribution. Using the implied returns from CAPM as the prior and then adding the investors' views, a posterior distribution can be obtained. The

following equation details on how to derive the implied returns,  $\Pi$ , of the assets in a portfolio. Assuming there are N-assets in the portfolio,  $\Pi$  will be a Nx1 vector.

$$(1) \quad \Pi = \delta \Sigma \omega$$

- $\delta =$  Risk aversion coefficient. It can either be an arbitrary assumption or can be given by  $\delta = (E(r) - r_f) / \sigma^2$
- $E(r) =$  Return of the market portfolio (a portfolio that includes all the assets in the market or any other index benchmark that the investor decide to choose)
- $r_f =$  Risk free market rate
- $\sigma^2 =$  Variance of the market portfolio
- $\Sigma =$  A covariance matrix of the assets (NxN matrix)
- $\omega =$  Weights of assets according to their market capitalization

After deriving the assets' implied returns, then we can compute the expected return,  $E(R)$  which is an Nx1 vector, of the assets under the Black-Litterman model with the following equation.

$$(2) \quad E(R) = [(\tau \Sigma)^{-1} + P^T \Omega P]^{-1} [(\tau \Sigma)^{-1} \Pi + P^T \Omega Q]$$

- $\tau =$  A scalar number indicating the uncertainty of the CAPM distribution (It is usually within the range of 0.025-0.05)
- $P =$  A matrix with investors views; each row a specific view of the market and each entry of the row represents the weights of each assets (KxN matrix)
- $Q =$  The expected returns of the portfolios from the views described in matrix P (Kx1 vector)
- $\Omega =$  A diagonal covariance matrix with entries of the uncertainty within each view (KxK matrix)
- $\Sigma$  and  $\Pi$  as described in previous section.

Below is an example on how to derive matrix P, Q, and  $\Omega$  based on specific investor views. Let there be 3 assets in the portfolio (A, B, and C), and the investors have 2 views about the market. First, Asset A will outperform all other assets by 5%. Second, Asset B will outperform asset C by 1%.  $\Omega$  is the covariance matrix between the two views where the investors specify their confidence level on their views. We derive the following matrices.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad Q = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \Omega = \begin{bmatrix} \omega 1 & 0 \\ 0 & \omega 2 \end{bmatrix}$$

Once we compute  $E(R)$ , then we can compute the posterior variance matrix, M, which will be used to compute the new covariance matrix. The new covariance matrix (equation 4) takes into account of the additional variance resulting from the investor views.

$$(3) \quad M = [(\tau \Sigma)^{-1} + P^T \Omega^{-1} P]^{-1}$$

$$(4) \quad \Sigma_p = \Sigma + M$$

With the new covariance matrix, we can then calculate the new portfolio weights, using equation (1) with new  $\Sigma_p$  and solving for  $\omega$  instead of  $\Pi$  this time<sup>1</sup>.

The following is a specific example illustrating the benefits of the Black-Litterman model. In our example regarding the international equity market, data has been provided for the index volatility, equilibrium portfolio weight, and implied returns. The weighting is based on the market capitalization of each market and the expected returns are implied by the market prices. Additionally, we found the covariance matrix for this dataset.

Country	Equity Index Volatility (%)	Equilibrium Portfolio Weight (%)	Equilibrium Expected Returns (%)
Australia	16.0	1.6	3.9
Canada	20.3	2.2	6.9
France	24.8	5.2	8.4
Germany	27.1	5.5	9.0
Japan	21.0	11.6	4.3
UK	20.0	12.4	6.8
USA	18.7	61.5	7.6

### Covariance Matrix

	AUS	CAN	FRA	GER	JAP	UK	USA
AUS	0.0256	0.01585	0.018967	0.02233	0.01475	0.016384	0.014691
CAN	0.01585	0.041209	0.033428	0.036034	0.027923	0.024685	0.024751
FRA	0.018967	0.033428	0.061504	0.057866	0.018488	0.038837	0.030979
GER	0.02233	0.036034	0.057866	0.073441	0.020146	0.042113	0.033092
JAP	0.01475	0.013215	0.018488	0.020146	0.0441	0.01701	0.012017
UK	0.016384	0.024685	0.038837	0.042113	0.01701	0.04	0.024385
USA	0.014691	0.029572	0.030979	0.033092	0.012017	0.024385	0.034969

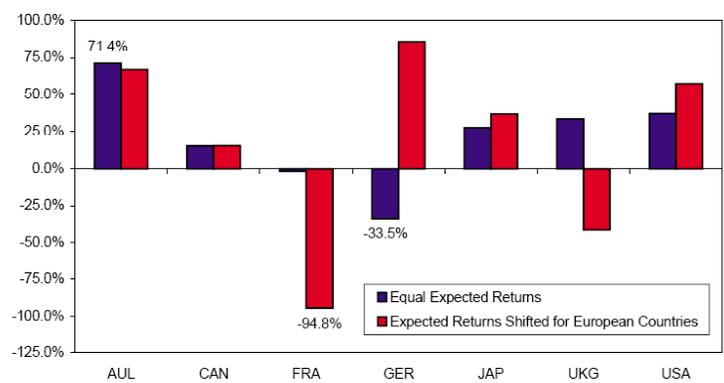
In our project, we simulated the results demonstrated by the Goldman Sachs paper. To begin the Black-Litterman framework, we use the implied market returns as a starting point for our expected returns. Then we are able to incorporate a view into our framework, with a corresponding confidence level. Versus the standard Markowitz model, where slight changes in expected returns may result in drastic changes in the portfolio allocation, the Black-Litterman model provides more intuitive and reasonable shifts in allocation.

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<sup>1</sup> Solving for  $\omega$ , formula becomes  $\omega = (\delta \Sigma_p)^{-1} \Pi$

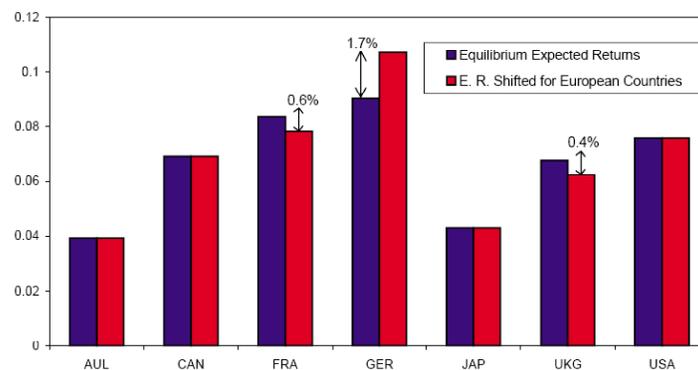
In our example, the German equity market is expected to outperform France and the UK by 5%. In the standard model, we start off with equal expected returns and specify the precise performance of those markets. Such a change would result in a large shift in weighting, as demonstrated in the figure.

### Portfolio Allocation using traditional approach and equal expected returns



Notice how the portfolio results in huge swings in position, including shifts in the USA, AUL, and JAP markets, even though the investor has no specific views about those markets. We can use the traditional model and start with equilibrium returns, but the traditional model specifies the changes in the German and European markets as well. This results in no changes in the others. The changes are based on the market-weighted expected returns.

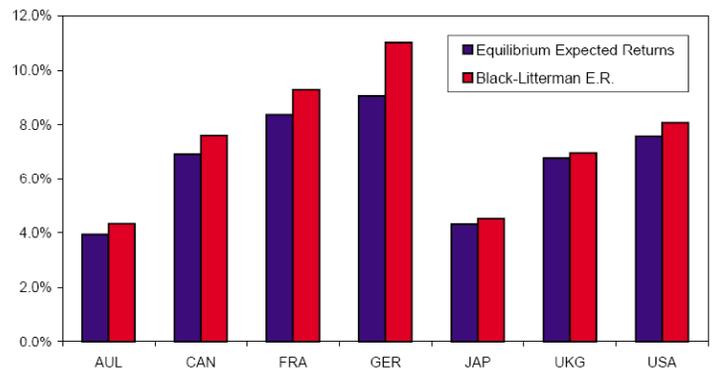
### Expected Returns using traditional model with equilibrium returns



With the Black-Litterman model, the first difference is that we specify the implied returns as the starting point. Secondly, you are able to express either a relative or absolute view through

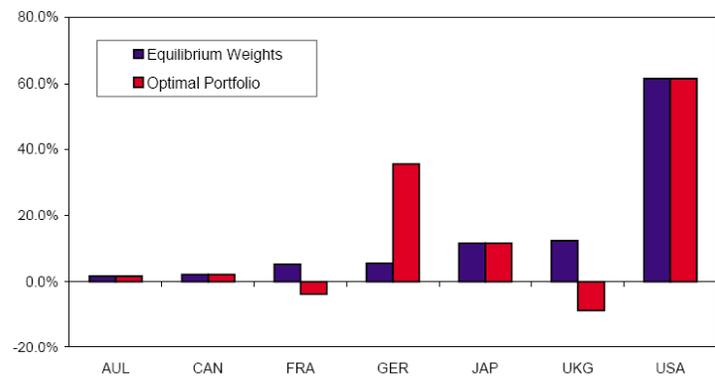
the P matrix. We can say that we believe Germany will be up 5% relative to UK and France, but the model does not require us to specify the exact percentages.

### Expected Returns using the Black-Litterman model



In the figure above, Germany outperforms France and the UK. Notice that the new expected returns are higher across the board; this is reasonable since the other markets are positively correlated with Germany. The chart below illustrates one of the main features of the model – notice how the weights of the markets on which investor has no view on is unchanged! It is only in the German, French, and British market where investor has a different position reflecting the view. The intuitive explanation is that the investor holds the market portfolio and then holds a portfolio of assets for which there is a view. This is different from the results in the standard Markowitz model.

### Optimal Portfolio Weights using the Black-Litterman model



This one view example is the simplest illustration of the Black-Litterman model. It can be extended, since the model also has as an input the variance in the views. Each view is assumed to

be independent but the variance of each view indicates the investor's confidence. In our case, if the confidence level is lower, the expected returns will also be lower since the investor is less certain about the rise in returns. The investor could likewise specify additional views, which would be incorporated in the weights of each asset. Note that the graphs are taken from the Goldman Sachs paper describing the model. Our code was able to generate the appropriate implied returns and expected returns given investor views, but we ran into problems when displaying the portfolio weights.

The Black-Litterman model provides an excellent tool for the investors and has an advantage over many traditional asset allocation models. The traditional models require investors to provide expected returns for all the assets in the portfolio. Investors often specialize in certain areas and cannot develop knowledgeable expected returns for all the diversified assets in the portfolio. Models using historical returns are not very useful because historical returns are often poor indicators of how the asset will perform in the future if it is very volatile. Some models do not consider the volatility of the assets or assume equal volatility among assets. This is poor practice because when given two assets with equivalent expected returns, investors would pick the asset with lower volatility. Other models fail to account for correlation between assets. The Markowitz model, in particular, is highly sensitive to any input changes, which often results in insensible portfolio weights. A slight adjustment in implied return in one or two assets will have large impact on the portfolio weights of all the assets in the portfolio, which is often not easily understandable by investors. Therefore, asset managers do not use these traditional allocation models as often as they should. Compared to the traditional models, Black-Litterman model allows investors to not only insert their own views on the market, but also weighs in the investor's level of confidence in their assumptions. For example, if the investors specialize in certain markets, then they can give their views in these assets with a much higher confidence than the others. This model has the ability to place more emphasis on stronger views to generate optimal portfolio weights.

Black-Litterman provides a powerful tool to investors if they have specific views about the market. One major drawback of this model is that it will not add much insight to the traditional asset allocation models unless the investors have preexisting opinions about the market. As the model gains more popularity within the industry, other researchers such as Idzorek, Satchell, and Scowcroft have, have further developed variations of this model, resulting

in multiple assumptions and methodology to this model. In addition, there are some extreme views on  $\tau$ , the uncertainty of CAPM distribution. Some believe that it should be closer to 0, but some argues that it is closer to 1. There are ongoing discussions on the value of  $\tau$ . Further readings is highly recommended for those are interested in further explore the derivation and the implementation of the Black-Litterman model.

### **Bibliography**

Black, F. and Litterman, R. (1991). "Global Asset Allocation with Equities, Bonds, and Currencies." Fixed Income Research, Goldman, Sachs & Company, October.

He, G. and Litterman, R. (1999). "The Intuition Behind Black-Litterman Model Portfolios." Investment Management Research, Goldman, Sachs & Company, December.

Black, Fischer and Robert Litterman, "Asset Allocation: Combining Investor Views With Market Equilibrium." Goldman, Sachs & Co., Fixed Income Research, September 1990.

Idzorek, Thomas M. "A Step-by-Step Guide to the Black-Litterman Model." Zehyr Associates, Inc. July, 2004.

Satchell, S. and Scowcroft, A. (2000). "A Demystification of the Black-Litterman Model: Managing Quantitative and Traditional Construction." Journal of Asset Management, September, 138-150.

## Appendix

```
#####
# Black-Litterman
#####

# Market parameters
# tau - uncertainty ratio
# delta - market risk aversion

# BL Inputs
# Pi - implied expected returns, nx1
# Sigma - covariance matrix of expected returns, nxn

# P - views matrix, kxn
# Omega - covariance matrix of views, kxk
# Q - relative return in view, kx1

BL<-function(Pi, Sigma, P, Omega, Q, tau, delta) {

#      E_r<-solve(solve(tau*Sigma)+t(P)**%Omega**P)**(solve(tau*Sigma)**%Pi+t(P)**%Omega**Q)

      E_r<-Pi+tau*Sigma**t(P)**solve((tau*P**Sigma**t(P))+Omega)**(Q-P**Pi)

#      M<-solve(solve(tau*Sigma)+t(P)**%solve(Omega)**%P)
      M<-tau*Sigma-tau*Sigma**t(P)**solve(tau*P**Sigma**t(P)+Omega)**(tau*P)**Sigma

      Sigma_p<-Sigma+M

      weights<-solve(delta*Sigma_p)**Pi
      weights
}

#####
# Portfolio optimizer
#####

# mu - means of log returns
# sigma - covariance matrix of log returns
# mustar - desired log returns

portopt<-function(mu, sigma, mustar) {

      I=matrix(1,nrow=length(sigma[,2]),ncol=1)
      A=as.numeric(t(mu)**%solve(sigma)**%I)
      B=as.numeric(t(mu)**%solve(sigma)**%mu)
      C=as.numeric(t(I)**%solve(sigma)**%I)
      D=as.numeric(B**%C-(A**%A))

      #Efficient Weights
      weights_p = (B*solve(sigma)**%I-A*solve(sigma)**%mu+mustar*(C*solve(sigma)**%mu-
A*solve(sigma)**%I))/D
      colnames(weights_p)="weights"

      weights_p
}

#####
# Sample data
#####
```

```

data<-cbind(c(.16, .203, .248, .271, .21, .2, .187), c(.016, .022, .052, .055, .116, .124, .615),
c(.039, .069, .084, .09, .043, .068, .076))
rownames(data)=c("AUS", "CAN", "FRA", "GER", "JAP", "UK", "USA")
colnames(data)=c("Vol", "Weight", "Returns")

cov<-read.csv("cov.csv")
rownames(cov)<-cov[,1]
cov<-cov[,-1]

tau<-.007
delta<-2.5

Pi<-as.matrix(data[, "Returns"])
Sigma<-as.matrix(cov)

P<-matrix(c(0,0,-data[,"Weight"]["UK"]/(data[,"Weight"]["UK"]+data[,"Weight"]["FRA"]),1,0,-
data[,"Weight"]["FRA"]/(data[,"Weight"]["UK"]+data[,"Weight"]["FRA"]),0), nrow=1)
Omega<-.00003
Q<-as.matrix(.05)

barplot(as.matrix(test*100), main="Market Sectors", ylab= "Percent Allocation",
beside=TRUE, col=rainbow(2))

# Place the legend at the top-left corner with no frame
# using rainbow colors
legend("topleft", c("ER Using BL", "ER"), cex=0.6,
bty="n", fill=rainbow(2));

```