# Social Choice and Social Networks

# Consensus, Bribe and Marketing

# Draft - All right reserved

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# Admin tasks of the day

- Scribe notes
- To get a grade you <u>must</u> do one.
-
- Today?
- Last week?
- Previous notes?

# Today topics

- Topic 1:
- Randomized consensus protocols
- The “voter model” and generalizations.

- Topic 2:
- Which voters to “buy” in the voter model.
- The viral marketing problem.

- Next week/month: a new topic - unbiased signals and social choice.

# Randomized Consensus protocols

- As before we will consider  a social network which is a graph.
- G=(V,E)

- We will work in continuous time and asynchronous fashion.
- At time 0 people hold opinions $X_0(v)$, $v \in V$.

- In vertex models:
- Update times of each vertex is a Poisson(1) process.
- Equivalent to:  after each update – waiting Exp(1) time.
- At update time: update $X_t(v)$ according to neighbors, current opinion  and <u>randomness</u>

- Edge models similar:
- Edges (u,v) have update times – both end points of edge update $X_u(t)$ and $X_v(t)$

# Randomized Consensus protocols

- Vertex models:

Individual decisions.

- Edge models:

Decisions are result of interactions with models.

- Natural to consider also clique models.

- “Goal”: find “good models” that converge to consensus.

What are good models?

# Good consensus models

- “Goal”: find “good models” that converge to consensus.
- Property 1: Fairness with respect to alternatives.
- Property 2: Consensus is a fixed point.
- Property 3: Simple.

- Many of the following slides due to G. Schoenebeck

# Big Question/General Goals


OR


Courtesy of
G. Schoenebeck

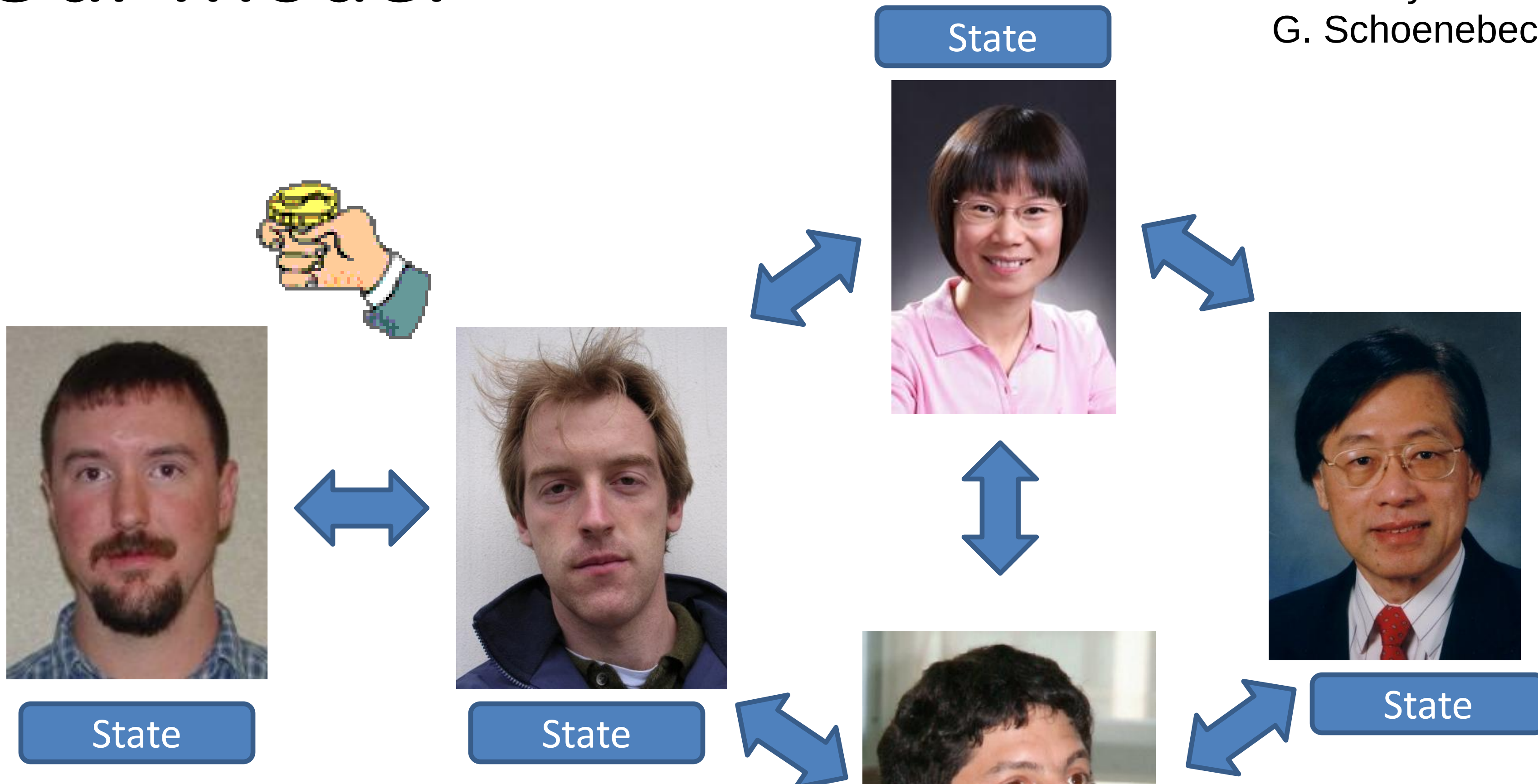
# Series of Experimental Work

- Latane L'Herrou[96]
  - Try to play majority
- Kearns, Judd, Tan, Wortmann [09]
  - Consensus with different payoffs
- Kearns, Suri, Montefort [06]
  - coloring
  - Enemark, McCubbins, Paturi [09]

Courtesy of
G. Schoenebeck

# Our Model

Courtesy of
G. Schoenebeck





$$T = T'(S_{Casey}, S_{Grant}, R, Advice) \times T'(S_{Grant}, S_{Casey}, R, Advice)$$

# Problems Studied

- Coordination : Arrive at consensus

- Majority Coordination:
- Arrive at consensus which equals majority of original opinions.

- *Protocols have to be symmetric with respect to the two states.*

Courtesy of
G. Schoenebeck

# Definitions: Broadcast and Collision time

- Broadcast Time:  Time for a message to flood network.
  - More like Expansion than Diameter
- Collision Time:  Time until for every pair of people, someone has received both of their messages.
  - Provides trivial lower bound

Courtesy of
G. Schoenebeck

# Related Work

- Similar to Distributed Computing
  - Usually a different time metric
    - Synchronous
    - Worst case
  - Usually different symmetry condition
- Coordination Games in Economics
- Similar to Simulations in Social Networking literature.
- More to come in context

Courtesy of
G. Schoenebeck

# Coordination using the Voter Model

- Voter Model is an edge model where:
- If $X_v(t-) = X_u(t-)$ then: $X_v(t) = X_u(t) = X_v(t-) = X_u(t-)$
- If $X_v(t-) \neq X_u(t-)$ then:
  - Prob ½: $X_v(t) = X_v(t-) \neq X_u(t) = X_u(t-)$
  - Prob ¼: $X_v(t) = X_u(t) = X_u(t-)$
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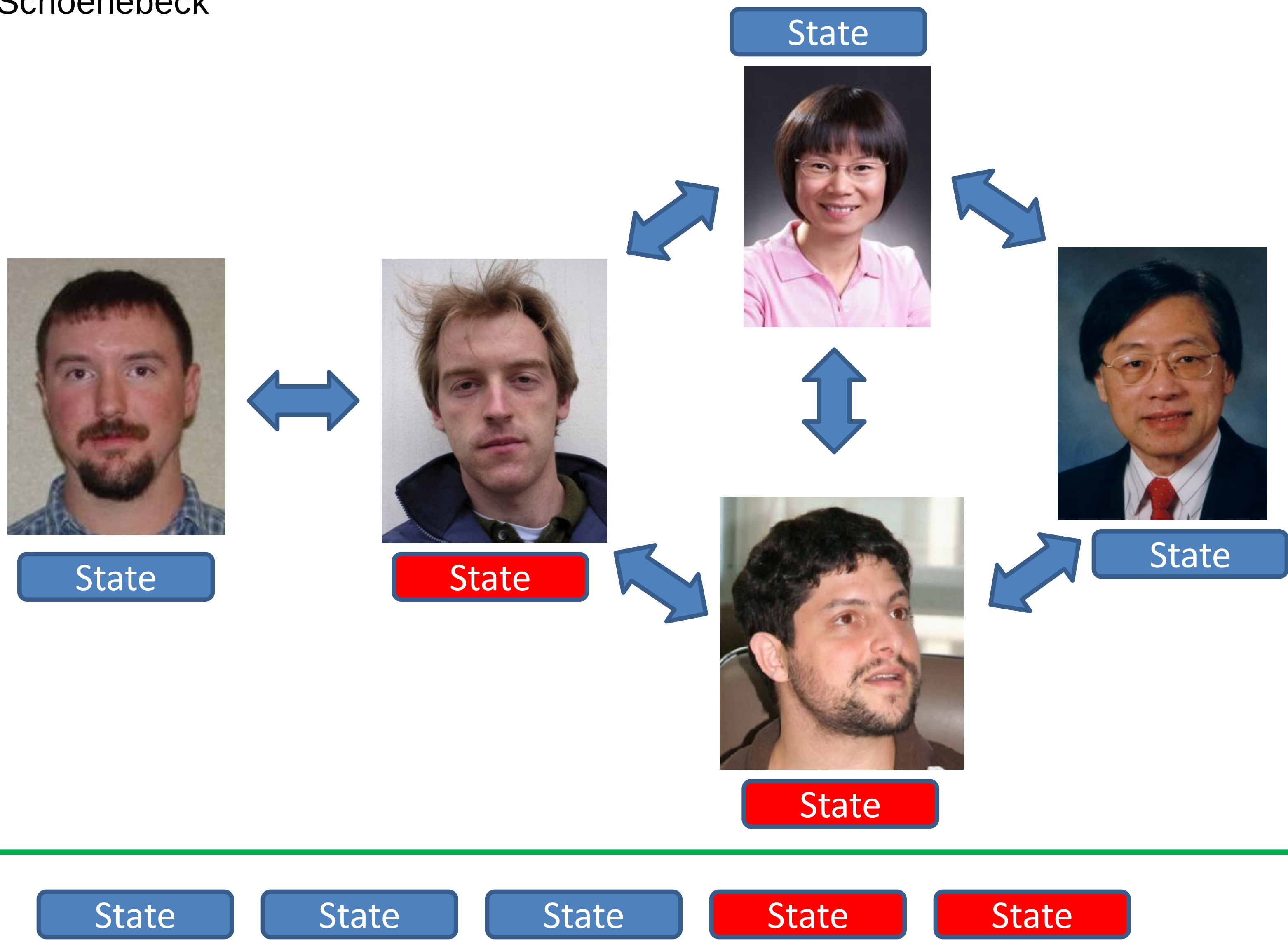
- If $X_v(t-) = X_u(t-)$ then: $X_v(t) = X_u(t) = X_v(t-) = X_u(t-)$
- P. Cliford and A. Sudury. A model for spatial conflict (1973) + Liggett
- A. Holley and T. M. Liggett. Ergodic theorems
- for weakly interacting infinite systems and the voter
- Model (1975)

# Voter Model

Courtesy of
G. Schoenebeck



# Convergence of the voter model

Claim: The voter model converges to consensus.

Moreover, the convergence time is $O(|V|^2)$.

Pf: ??

Courtesy of
G. Schoenebeck

# Convergence of the voter model

Claim: The voter model converges to consensus.

Moreover, the convergence time is $O(|V|^2)$.

<u>Pf of Convergence:</u>

Each of the consensus configurations is a fixed point. It is also clearly reachable from any other configurations.

<u>Pf idea of Convergence:</u>

Let opinions be +,-. Then $X(t) = \sum X_v(t)$ is a martingale.

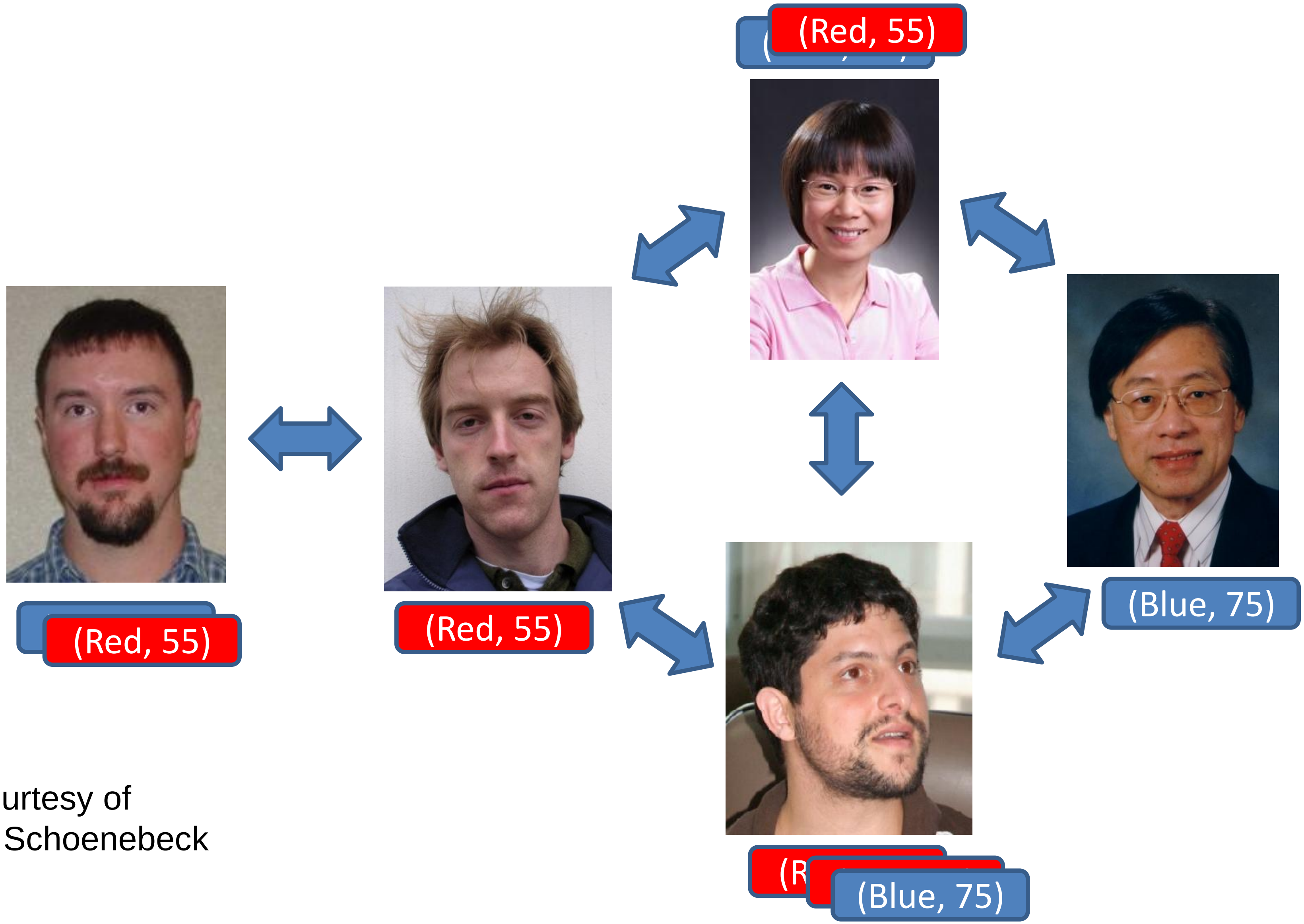
# Convergence of the voter model

- <u>Convergence in terms of expansion:</u>
- Let r = smallest cut in the graph. Then for any starting configuration will converge to consensus by time $n^2/r$ with probability at least ½.
- <u>Pf:</u> X(t) is a martingale. Let P(t) = P(Cons. by t).
- f(t) = Var[X(t)] satisfies:
- $f'(t) = \lim E[(X(t+h)-X(t))^2]/h$
- $f'(t) \geq r\, P(\text{No consensus at time } t)$
- If $P(4\, n^2 / r) \leq ½$ then
- $n^2 \geq f(4\, n^2/ r) \geq ½\ r\, 4\, n^2 / r$ – contradiction.

# Coordination by choosing a leader

- Each player chooses at random how strong $S(v)$ her opinion is in the range $[n^{10}]$.
- $X_v(0+) = X_v(0) \times S(v)$.
- In edge update: weak copies strong.
- In case of tie – voter like dynamics.

# Greatest Element Dynamics

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# Coordination by choosing a leader

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- In case of tie – voter like dynamics.
- Analysis: in case of single leader – broadcast time.
- In the case of more than one – like voter model.
- Expected time := broadcast time.

# Coordination Summary

| Problem | Memory | Time | Required Advice |
|---|---|---|---|
| Voter Model | 1 | $n^2$ | none |
| Greatest Element | $O(\log(n))$ | broadcast | $\Theta(\log(\|V\|))$ |
| Wait-and-See | expected $O(1)$ | $O(\text{broadcast})$ | $\Theta(\text{broadcast} \cdot \|E\|)$ |

Courtesy of
G. Schoenebeck

# The Majority Coordination problem

- Claim: Cannot be done with no memory.
- Pf: ??

# The Majority Coordination problem

- Claim: Cannot be done with no memory.
- Pf: Look at a configuration resulting in a change.
- Make it change majority.

# The Majority Coordination problem

- Claim: Can be done with 2 bits of memory.
- Pf: ??

# Strong Weak Voter

- All Voters have **opinion** (red/blue) and **strength** of opinion (STRONG/weak). Originally all strong.
- When they meet,
  - Update color:
    - STRONG influence weak
    - Otherwise voter model
  - Update Strengths:
    - Two STRONGS of different colors cancel to weak
    - Otherwise stay the same
    - STRONG/weak swap strengths

Courtesy of
G. Schoenebeck

# Majority Coordination

| | Memory | Time | Required Advice |
|---|---|---|---|
| [LB95] | 1 | impossible | |
| [BTV09] | 2 | $< \infty$ | none |
| Strong-Weak | 2 | $O(n^3)$ | none |
| [KT08] | $O(\log(n))$ | $O(n^7)$ | $\lvert V \rvert$ |
| Wait-and-See | expected $O(\log(\Delta))$ | $O(d + \log(n)) \cdot \log(n)$ | $\Theta(\text{broadcast} \cdot \lvert E \rvert)$ |

Courtesy of
G. Schoenebeck

# Next topic: which voters to buy

- Q: Consider the voter model on a graph.
- Suppose you can change the opinions of r people from – to +.
- Which opinions should you change?
- Want to maximize the probability that convergence to all +.

# Which voters to buy – a different model

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- Suppose you can change the opinions of r people from – to +.
- Which opinions should you change?
- Want to maximize the probability that convergence to all +.

- A: It doesn't matter since it is the same Random walk.

- Consider a synchronous model where
- $X_v(t+1) = +/-$ with probabilities $\# \{w \in N(v) : X_w(t) = +/-\}$

- Who should we choose here?
- Difference between vertex model and edge model.
- Question asked by: Even Dar and Shapira

# Which voters to buy – a different model

- Consider a synchronous model where
- $X_v(t+1) = +/-$ with probabilities $\# \{w \in N(v) : X_w(t) = +/-\}$
- Want to change X_v(0) for s nodes to maximize probability of final vote to be +. Who should we choose?

- Who should we choose here?
- Difference between vertex model and edge model.

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- Who should we choose here?
- Claim: we should choose the highest degree nodes that are-.
- Claim: $\sum_v d_v X_v(t)$ is a martingale.

- This problem leads us naturally to other problems involving changing people opinions. Next we discuss the “Viral Marketing Problem”

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- This problem leads us naturally to other problems involving changing people opinions. Next we discuss the "Viral Marketing Problem"
- Most of the next slides are due to Sebastien Roch from a joint paper on viral marketing.

# Some Network Optimization Problems

- Problem:
  - Optimization over stochastic models defined on networks.

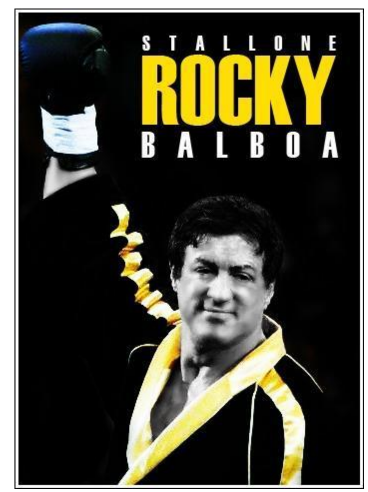
- Examples:
  - Which Genes to knock out in order to kill a cancer cell?
  - Which computers to immune in order make a networks robust?
  - Which computers to attack in order to fail the network?
  - Which individuals to immune to stop a disease from spreading.

  - Viral Marketing: Which individuals to expose to a product so as to maximize its distribution?

# models of collective behavior

- examples:
  - joining a riot
  - adopting a product
  - going to a movie

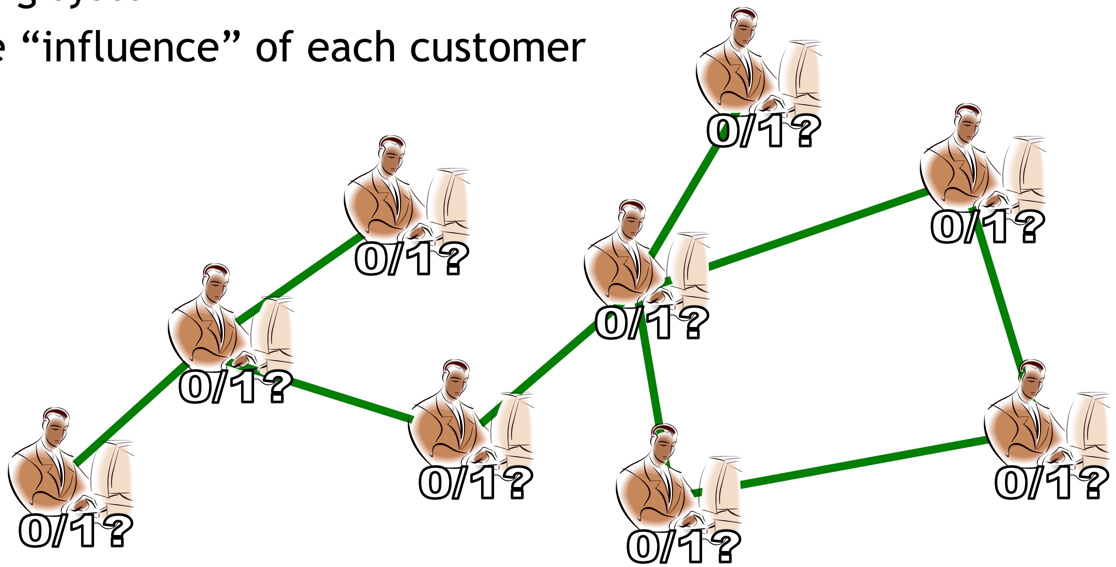
- model features:
  - binary decision
  - cascade effect
  - network structure


Courtesy of
S. Roch

# viral marketing

- referrals, word-of-mouth can be very effective
  - ex.: Hotmail
- viral marketing
  - goal: **mining the network value** of potential customers
  - how: **target a small set of trendsetters**, seeds
- example [Domingos-Richardson'02]
  - collaborative filtering system
  - use MRF to compute "influence" of each customer

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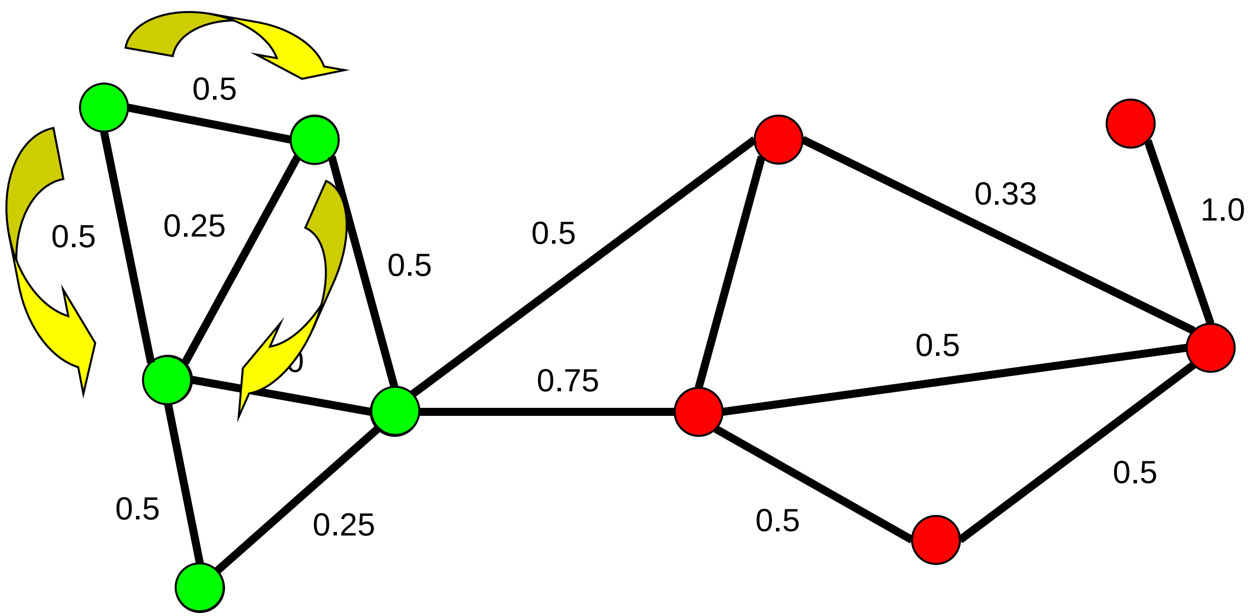
# independent cascade model

- when a node is activated
  - it gets one chance to activate each neighbour
  - probability of success from u to v is $p_{u,v}$

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# Independent Cascades Model

- graph $G=(V,E)$; initial activated set $S_0$, $S_1$ or $S_0 \cup S_1$, $S_0 \cap S_1$
- a,b and c,d the expected size of the marketed set given starting at the 4 sets then:
- Claim: c+d $\leq$ a+b
- Pf: ???

# Independent Cascades Model

- graph $G=(V,E)$; initial activated set $S_{0,}$ $S_1$ or $S_0 \cup S_1$, $S_0 \cap S_1$
- a,b and c the expected size of the marketed set given starting at the three sets then:
- <u>Claim:</u> c +d $\leq$ a+b
- <u>Pf:</u> Use the same randomness to decide which edges copy and which not.
- Let G' be the (random) induced graph.
- Then a = E[vertices connected to $S_0$ in G]
- And b = E[vertices connected to $S_1$ in G]
- And c = E[vertices connected to $S_0 \cup S_1$ in G]
- And d = E[vertices connected to $S_0 \cup S_1$ in G]

- This means that the expected size of the infected set is a submodular functions of the set.

# generalized models

- graph $G=(V,E)$; initial activated set $S_0$

- generalized threshold model [Kempe-Kleinberg-Tardos'03,'05]
  - activation functions: $f_u(S)$ where $S$ is set of activated nodes
  - threshold value: $\theta_u$ uniform in $[0,1]$
  - dynamics: at time $t$,set $S_t$ to $S_{t-1}$ and add all nodes with $f_u(S_{t-1}) \geq \theta_u$

    (note the process stops after (at most) $n-1$ steps)
- generalized cascade model [KKT'03,'05]
  - when node $u$ is activated:
    - gets one chance to activate each of the neighbours
    - probability of success from $u$ to $v$: $p_u(v,S)$ where $S$ is set of nodes who have already tried (and failed) to activate $u$
  - assumption: the $p_u(v,.)$'s are "order-independent"

- **theorem** [KKT'03] - the two models are equivalent

Courtesy of
S. Roch

# influence maximization

- **definition** - the **influence** $\sigma(S)$ given the initial seed $S$ is the expected size of the infected set at termination

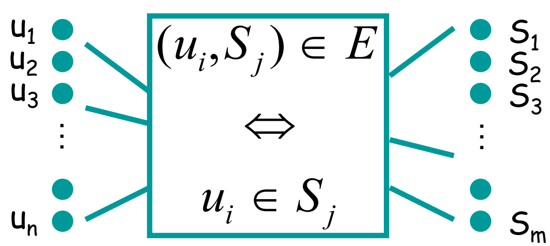
$$\sigma(S) = \mathrm{E}_S\left[\left|S_{n-1}\right|\right]$$

- **definition** - in the **influence maximization problem (IMP)**, we want to find the seed $S$ of fixed size $k$ that maximizes the influence

$$S^* = \arg\max\left\{\sigma(S) : S \subseteq V,\ |S| = k\right\}$$

- **theorem** [KKT'03] - the IMP is **NP-hard**
  - reduction from *Set Cover*: ground set $U = \{u_1,\ldots,u_n\}$ and collection of cover subsets $S_1,\ldots,S_m$

independent cascade model




$$\exists S,\ |S| = k,\ \sigma(S) \geq n + k\,?$$

Courtesy of S. Roch

# submodularity

- definition - a set function $f : V \to R$ is **submodular** if for all $A$, $B$ in $V$

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

- example: $f(S) = g(|S|)$ where $g$ is concave

- interpretation: “discrete concavity” or “diminishing returns”, indeed submodularity equivalent to

$$\forall S \subseteq T, \forall v \in V, \quad f(T \cup \{v\}) - f(T) \leq f(S \cup \{v\}) - f(S)$$

- threshold models:
  - it is natural to assume that the **activation functions have diminishing returns**
  - supported by observations of [Leskovec-Adamic-Huberman’06] in the context of viral marketing

Courtesy of S. Roch

# main result

- **theorem** [M-Roch'06; first conjectured in KKT'03] - in the generalized threshold model, if all activation functions are monotone and submodular, then the **influence is also submodular**

- **corollary** [M-Roch'06] - IMP admits a $(1 - e^{-1} - \varepsilon)$-approximation algorithm (for all $\varepsilon > 0$)
  - this follows from a general result on the approximation of submodular functions [Nemhauser-Wolsey-Fisher'78]

- known special cases [KKT'03,'05]:
  - linear threshold model, independent cascade model
  - decreasing cascade model, "normalized" submodular threshold model

$$\forall S \subseteq T,\ p_u(v,S) \geq p_u(v,T) \text{ or equiv. } \frac{f_u(S \cup \{v\}) - f_u(S)}{1 - f_u(S)} \geq \frac{f_u(T \cup \{v\}) - f_u(T)}{1 - f_u(T)}$$

Courtesy of
S. Roch

# Easy approximation of sub-modular functions

- **Thm:** Let f : $2^{[n]} \rightarrow [0,1]$ be monotone and submodular.
- Consider finding the set S maximizing f(S) under the constraints |S|=k.
- Then the greedy algorithm provides a (1-1/e) approximate solution to this problem.

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- Consider finding the set S maximizing f(S) under the constraints |S|=k.
- Then the greedy algorithm provides a $(1-1/e-\varepsilon)$ approximate solution to this problem.

- Pf:
- Let S_1,…S_k be the sets chosen by the greedy alg. O = optimal set.
- Write $x_i = f(S_i)-f(S_{i-1})$
- Then $f(O) \leq f(S_i \cup O) \leq f(S_i) + k\, x_{i+1}$
- So $x_{i+1} \geq (f(O)-f(S_i))/k$.
- By induction: $f(S_i) = f(S_{i-1}) + x_i \geq f(O)\ (1 - (1-1/k)^i)$
- Taking i=k we obtain the claim

# related work

- sociology
  - threshold models: [Granovetter'78], [Morris'00]
  - cascades: [Watts'02]

- data mining
  - viral marketing: [KKT'03,'05], [Domingos-Richardson'02]
  - recommendation networks: [Leskovec-Singh-Kleinberg'05], [Leskovec-Adamic-Huberman'06]

- economics
  - game-theoretic point of view: [Ellison'93], [Young'02]

- probability theory
  - Markov random fields, Glauber dynamics
  - percolation
  - interacting particle systems: voter model, contact process

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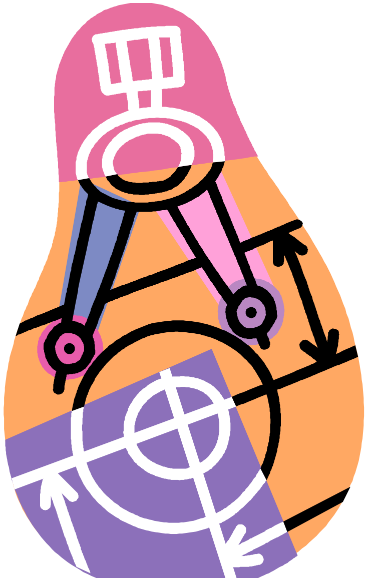
## proof sketch



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# coupling

- we use the generalized threshold model
- arbitrary sets $A$, $B$; consider 4 processes:
  - $(A_t)$ started at $A$
  - $(B_t)$ started at $B$
  - $(C_t)$ started at $A \cap B$
  - $(D_t)$ started at $A \cup B$
- it suffices to **couple the 4 processes** in such a way that for all $t$

$$C_t \subseteq A_t \cap B_t$$
$$D_t \subseteq A_t \cup B_t$$

- indeed, at termination

$$|A_{n-1}| + |B_{n-1}| \geq |A_{n-1} \cap B_{n-1}| + |A_{n-1} \cup B_{n-1}| \geq |C_{n-1}| + |D_{n-1}|$$

(note this works with $|.|$ replaced with any $w$ monotone, submodular)

Courtesy of
S. Roch

# proof ideas

- our goal:

$$C_t \subseteq A_t \cap B_t \quad (1) \qquad D_t \subseteq A_t \cup B_t \quad (2)$$

- antisense coupling
  - obvious way to couple: use same $\theta_u$'s for all 4 processes
  - satisfies (1) but not (2)
  - "antisense": using $\theta_u$ for $(A_t)$ and $(1-\theta_u)$ for $(B_t)$ "maximizes union"
  - we combine both couplings
- piecemeal growth
  - seed sets can be introduced in stages
  - we add $A \cap B$ then $A \setminus B$ and finally $B \setminus A$
- need-to-know
  - not necessary to pick all $\theta_u$'s at beginning
  - can unveil only what we need to know:

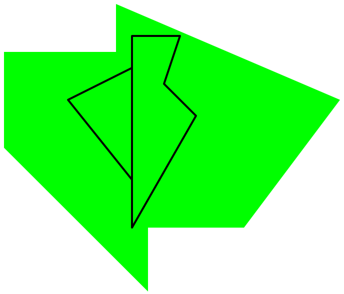
$$\theta_v \in [f_v(S_{t-2}), f_v(S_{t-1})]?$$

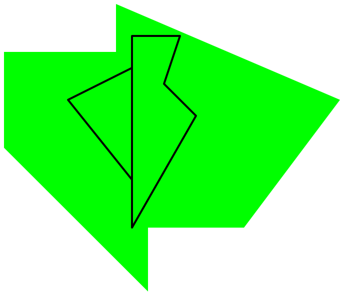
Courtesy of
S. Roch

# piecemeal growth

- process started at $S$: $(S_t)$
- **partition** of $S$: $S^{(1)},\ldots,S^{(K)}$
- consider the process $(T_t)$:
  - pick $\theta_u$'s
  - run the process with seed $S^{(1)}$ until termination
  - add $S^{(2)}$ and continue until termination
  - add $S^{(3)}$ and so on

- **lemma** - the sets $S_{n-1}$ and $T_{Kn-1}$ are the same distribution





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# antisense coupling

- disjoint sets: $S$, $T$
- partition of $S$: $S^{(1)},\ldots,S^{(K)}$
- piecemeal process with seeds $S^{(1)},\ldots,S^{(K)},T$: $(S_t)$

- consider the process $(T_t)$:
  - pick $\theta_u$'s
  - run piecemeal process with seeds $S^{(1)},\ldots,S^{(K)}$ until termination
  - add $T$ and continue with **threshold values**

$$\theta_v' = 1 - \theta_v + f_v\left(T_{Kn-1}\right)$$

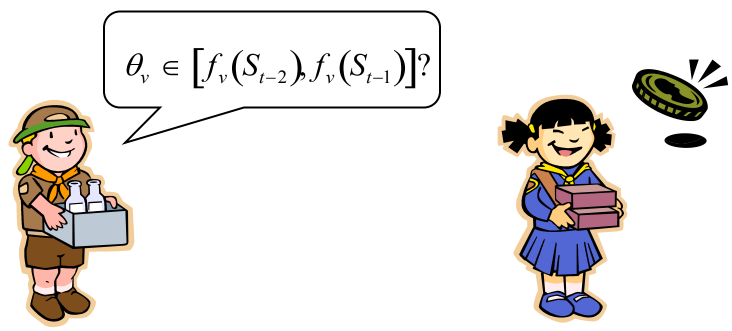
- **lemma** - the sets $S_{(K+1)n-1}$ and $T_{(K+1)n-1}$ have the same distribution

Courtesy of
S. Roch

# need-to-know

- proof of lemma
  - run the first K stages identically in both processes
  - note that for all v **not in** $S_{Kn-1} = T_{Kn-1}$, $\theta_v$ is uniformly distributed in $[f_v(T_{Kn-1}),1]$
  - but $\theta_v' = 1 - \theta_v + f_v(T_{Kn-1})$ has the same distribution

simulation 1




simulation 2

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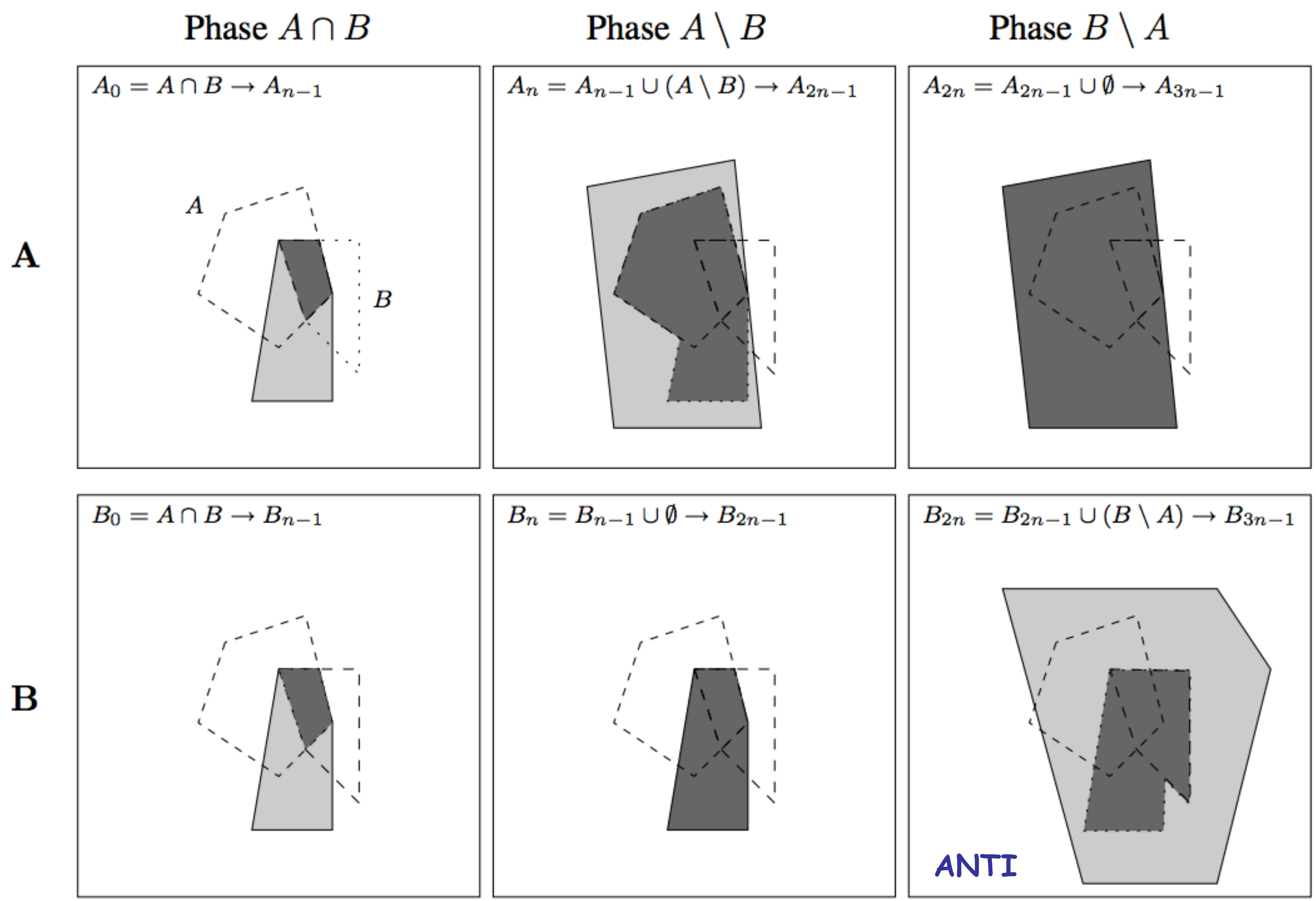
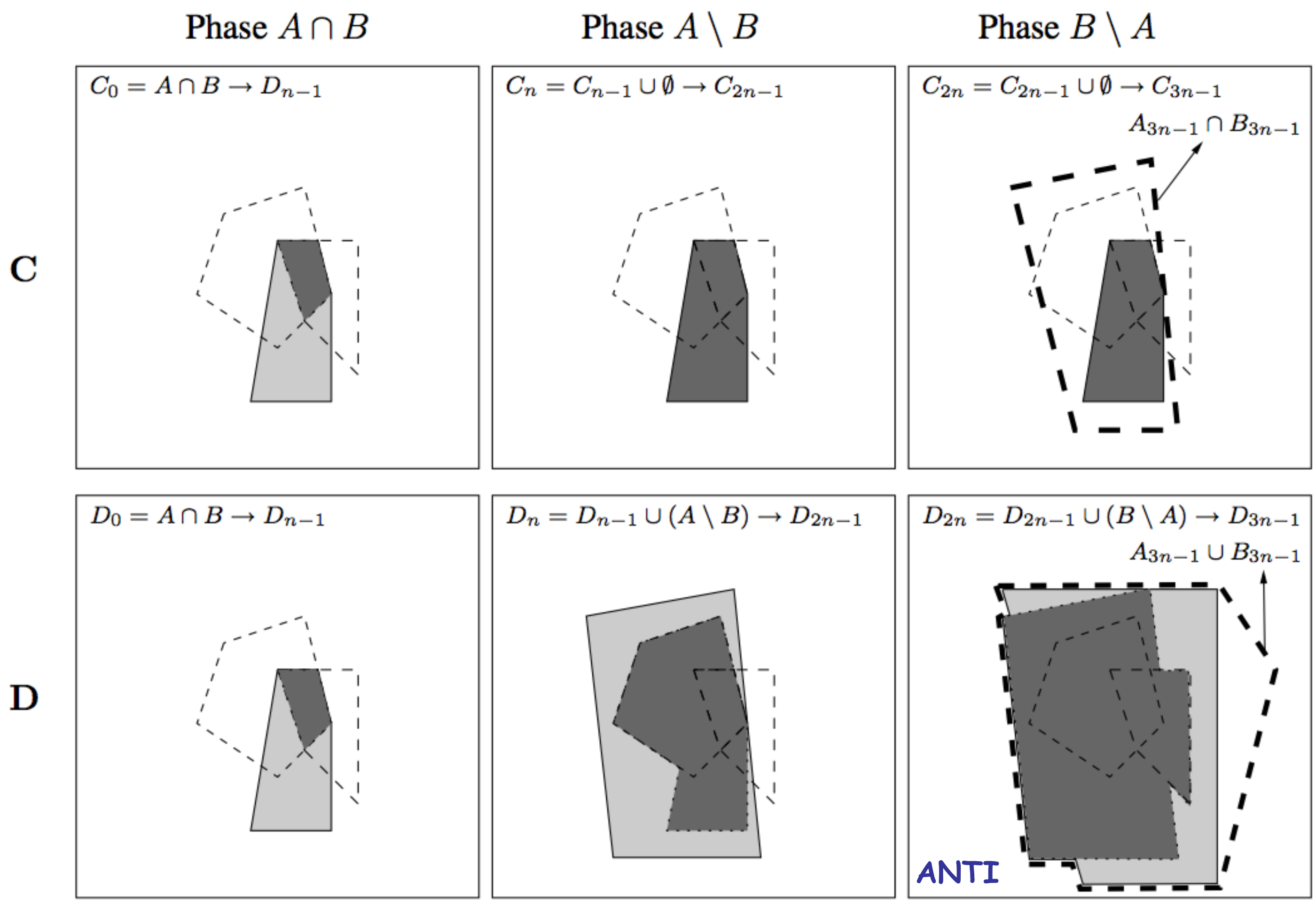
# Coupling proof I



# Coupling proof II



# Coupling proof III

- new processes have correct final distribution
- up to time 2n-1, $B_t = C_t$ and $A_t = D_t$ so that

$$C_t \subseteq A_t \cap B_t \qquad D_t \subseteq A_t \cup B_t$$

- for time 2n, note that

$$B_{2n-1} \subseteq D_{2n-1}$$

$$B_{2n} = B_{2n-1} \cup (T \setminus S) \qquad D_{2n} = D_{2n-1} \cup (T \setminus S)$$

- so by **monotonicity** and **submodularity**

$$f_v(B_{2n}) - f_v(B_{2n-1}) \geq f_v(D_{2n}) - f_v(D_{2n-1})$$

- then proceed by induction preserving

$$D_t \setminus D_{2n-1} \subseteq B_t \setminus B_{2n-1} \qquad \mathrm{f}_{\mathrm{v}}(D_t) - \mathrm{f}_{\mathrm{v}}(D_{2n-1}) \leq \mathrm{f}_{\mathrm{v}}(B_t) - \mathrm{f}_{\mathrm{v}}(B_{2n-1})$$

- At time t=3n-1, obtain $D_{3n-1} \subseteq D_{2n-1} \cup B_{3n-1} \subseteq A_{3n-1} \cup B_{3n-1}$

# general result

- we have proved:

  **theorem** [Mossel-R'06] - in the generalized threshold model, if all activation functions are submodular, then for any monotone, submodular function $w$, the generalized influence

  $$\sigma_w(S) = \mathrm{E}_S[w(S_{n-1})]$$

  is submodular
- Note: A closure property for sub-modular functions!

# Future Research Directions

- Study optimization problems for other stochastic models defined on networks.