Social Networks and Social Choice **Lecture Date: September 2, 2010**

Condorcet's theorem (continued)

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We concluded the last lecture by proving that the best aggregation function is majority vote. Next, we will look at the worst aggregation function.

1 The worst aggregation function

Proposition 1.1. *The worst fair, monotone aggregation function are the dictator functions* $x \mapsto x_i$.

To prove Proposition 1.1, we need to develop some results about the influence of single voters on the behavior of the aggregation function. First, we define a function that represents the influence of a single voter:

$$f_i(x) = f(x_1, \ldots, x_{i-1}, +, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, -, x_{i+1}, \ldots, x_n). \tag{1}$$

Recall that $C(f,p)$ is the probability that the aggregation function f reaches the correct conclusion when each voter independently comes to the correct answer with probability p. Then we can write the derivative of $C(f,p)$ in terms of the functions f_i:

Claim 1.2.

$$\frac{\partial}{\partial p} C(f,p) = \frac{1}{2} \sum_{i=1}^n \mathbb{E}_p f_i = \sum_{i=1}^n \frac{\mathrm{var}_{i,p}(f)}{4p(1-p)},$$

where

$$\mathrm{var}_{i,p}(f) = \mathbb{E}_p \, \mathrm{var}_p(f | x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Proof. Without loss of generality, let's condition on the event that the true signal is $+$. Note that if $x \in \{-1, 1\}$ then $\frac{\partial}{\partial p} p^{(1+x)/2}(1-p)^{(1-x)/2} = x$. Suppose we allow the voters to see the true signal with different probabilities and set $p_i = P(x_i = +)$. Then

$$\begin{aligned}
\frac{\partial C(f,p)}{\partial p_1} &= \frac{\partial}{\partial p_1} \sum_{x \in \{\pm 1\}^n} P(x) 1_{\{f(x)=+\}} \\
&= \frac{\partial}{\partial p_1} \sum_{x \in \{\pm 1\}^n} P(x) \frac{f(x)-1}{2} \\
&= \sum_{x \in \{\pm 1\}^n} P(x_2, \ldots, x_n) \frac{\partial P(x_1)}{\partial p_1} \frac{f(x)-1}{2} \\
&= \sum_{x \in \{\pm 1\}^n} P(x_2, \ldots, x_n) x_1 \frac{f(x)-1}{2} \\
&= \sum_{x_2, \ldots, x_n \in \{\pm 1\}} P(x_2, \ldots, x_n) \frac{f(+, x_2, \ldots, x_n) - f(-, x_2, \ldots, x_n)}{2} \\
&= \mathbb{E}_p \frac{f_1}{2}.
\end{aligned}$$

A similar computation works for $i \neq 1$ and hence,

$$\frac{\partial C(f,p)}{\partial p} = \sum_{i=1}^n \frac{\partial C(f,p)}{\partial p_i} = \frac{1}{2} \sum_{i=1}^n \mathbb{E}_p f_i.$$

For the second inequality, condition on $x_2, \ldots, x_n$ and note that

$$\operatorname{var}_p(f | x_2, \ldots, x_n) = \begin{cases} 4p(1-p) & \text{if } f_1(x_2, \ldots, x_n) = 2 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\mathbb{E} \operatorname{var}_p(f | x_2, \ldots, x_n) = 2p(1-p)\mathbb{E}_p f_1.$$

Repeating this for $i \neq 1$ yields the second equality. □

We will also use the following Efron-Stein-type inequality:

Claim 1.3. *If f is a function of n variables, then*

$$\operatorname{var}(f) \leq \sum_{i=1}^n \operatorname{var}_i(f),$$

where $\operatorname{var}_i(f) = \mathbb{E} \operatorname{var}(f(x) | x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Exercise 1.4. *Prove Claim 1.3. Two possible proofs involve Fourier analysis or a decomposition of variance by martingale differences.*

Now we are ready to prove that the worst aggregation function is a dictatorship.

Proof of Proposition 1.1. First of all, note that for all fair, monotone functions g, $C(g, 0.5) = 0.5$ and $C(g, 1) = 1$. Now, let f be a dictator function and suppose, for a contradiction, that f is not the worst aggregation function. That is, suppose there is some function g and some $p > 1/2$ such that $C(f, p) > C(g, p)$. Let $q = inf\{p : C(f, p) > C(g, p)\}$. Since $C(\cdot, p)$ is continuously differentiable with respect to p, $C(f, q) = C(g, q)$ and $\frac{\partial}{\partial q}C(f, q) \geq \frac{\partial}{\partial q}C(g, q)$. Thus,

$$\mathrm{var}_q(g) = \mathrm{var}_q(f) = \sum_i \mathrm{var}_{i,q}(f) \geq \sum_i \mathrm{var}_{i,q}(g)$$

where the first equality holds because $C(f, q) = C(g, q)$, the second equality holds because f is a function of only one variable and the inequality comes from Claim 1.2 and the fact that $\frac{\partial}{\partial q}C(f, q) \geq \frac{\partial}{\partial q}C(g, q)$. Looking at the two ends of the above inequality, we conclude (by Claim 1.3) that g is a function of one variable. Since g is fair and monotone, it must be a dictator function also. □

2 Other aggregation functions

So far, we have only seen 3 examples of aggregation functions. We saw that a simple majority is the best aggregation function and that dictatorship is the worst (at least, when we restrict to "reasonable" aggregation functions) and that it doesn't aggregate at all. We saw that a simplified "electoral college" aggregation aggregates well, and that it performs at the same asymptotic rate as simple majority (that is, it aggregates provided that $\sqrt{n}(p - 1/2)$ tends to infinity). Let's consider two more examples:

Example 2.1. *Given a number of "levels", k say, we can define a* recursive majority function *by drawing a tree of depth k and assigning a voter to each leaf. The vote is determined by proagating majority votes upward. That is, each node determines its vote by taking the majority of its children. For example, each city determines its vote by the majority of people living in that city, each county determines its vote by allocating a single vote to each city within it and taking the majority, and so on. Apparently, the USSR has such a system with up to 5 levels.*

It's not hard to see that this is different from simple majority voting. In fact, it is a generalization of the electoral college. If k is fixed, it aggregates at the same rate as the simple majority.

Example 2.2. *Suppose we have three judges. If they come to a unanimous conclusion, then that is the final decision. If not, we add two more judges and try again. The process will stop when there are n judges, in which case we revert to a simple majority if they still can't agree. This decision procedure is called "extensive forum" and it doesn't aggregate: there is* *a* $(1-p)^3$ *chance that the first three judges will get it wrong straight away, regardless of how large* n *is.*

3 The effect of a voter

It turns out that there are some general techniques, based on quantifying the influence of a single voter, for studying aggregation functions.

Definition 3.1. *The* influence of voter i *is* $\mathbb{E}_p f_i$.

The following theorem was due to Talagrand in 1994 and it says, roughly speaking, that if no voter has a large influence then an aggregation function performs well.

Theorem 3.2 (Talagrand [3])**.** *Let* f *be a monotone function and suppose that* $p < q$*. If* $\delta = \max_r \max_i \mathbb{E}_r f_i$ *then*

$$\mathbb{E}_p(f|s=+)(1-\mathbb{E}_q(f|s=+)) \leq \exp(c(q-p)\ln\delta)$$

for an absolute constant $c > 0$.

In particular, if f is fair and monotone and we set $p = 1/2$ in the above theorem, then $\mathbb{E}_p(f|s=+) = 1/2$ and so the probability of aggregating correctly is at least $1-\exp(c'(q-1/2)\ln\delta)$. If δ is very small, then the exponent is very large and negative, so we aggregate well.

An important special case was studied by Friedgut and Kalai [1]. They considered the case where f is invariant under the action of a transitive group G on $[n]$, in which case we say that f is *transitive*. (Recall that a group acting on $[n]$ is said to act *transitively* if for every $i,j \leq n$ there exists some $\sigma \in G$ such that $\sigma(i) = j$. Saying that f in *invariant* under this action means that $f(x) = f(x_\sigma)$, where $x_\sigma(i) = x(\sigma(i))$, for every $\sigma \in G$.) This says that each voter is "equal" in some sense, or at least that no voter is more powerful than any other voter.

Theorem 3.3. *If* f *is transitive and monotone and* $\mathbb{E}_p(f|s=+) > \epsilon$ *then* $\mathbb{E}_q(f|s=+) > 1-\epsilon$*, where*

$$q = p + c\frac{\log(1/(2\epsilon))}{\log n}.$$

If we assume, additionally, that f is fair, then we can set $p = 1/2$ in the above theorem and conclude that the aggregation function is correct with probability at least $1-\epsilon$ and $q = 1/2 + x\log(1/(2\epsilon))/(\log n)$. In other words, aggregation works provided that each signal is correct with probability of the order $1/2 + c\log^{-1} n$. This is not as sharp as the threshold of $1/2 + cn^{-1/2}$ that is provided by simple majority and friends, but it still allows us to send the quality of a single voter to zero.

4 Non-independent signals

All of the previous material assumed that the voters chose their votes independently. Of course, we would not expect this to be true in real life, so we would like to discuss the possibility of dependent signals. As before, we will assume that each voter receives the correct signal with probability at least p. However, we will be switching notation (so as not to deviate too far from the PowerPoint slides) so that the correct signal is always 1 and the incorrect signal is 0.

Clearly the Condorcet jury theorem cannot hold in general for the case of dependent voters. For example, we could assume that all the voters give the same vote: 1 with probability p and 0 otherwise. Then any fair, monotone aggregation function will have probability p of getting the correct answer, no matter what n is. Motivated by this example, we would like to have some parameter which quantifies the dependence of the outcome on a single voter i.

Definition 4.1. *The* effect of voter i *on the aggregation function* $f : \{0,1\}^n \to \{0,1\}$ *for a probability distribution* $\mathbb{P}$ *is*

$$e_i(f, \mathbb{P}) = \mathbb{E}(f|X_i = 1) - \mathbb{E}(f|X_i = 0).$$

If we set $\mathbb{E}X_i = p$, we have

$$\mathrm{cov}(f, X_i) = \mathbb{E}(f \cdot (X_i - p)) = p\mathbb{E}((1-p)f|X_i = 1) + (1-p)\mathbb{E}(-pf|X_i = 0) = p(1-p)e_i(f, P).$$

Theorem 4.2 (Haggstrom, Kalai, Mossel [2])**.** *Suppose that* $P(X_i = 1) \geq p > 1/2$ *for all* i*. Let* f *be the simple majority function and assume that* $e_i(f, \mathbb{P}) \leq \epsilon$ *for all* i*. Then majority aggregates correctly with probability at least* $1 - c\epsilon/(p - 1/2)$*, for some constant* $c > 0$*.*

Proof. Let $p_i = \mathbb{E}X_i$ and set $Y_i = p_i - X_i$ and $g = 1 - f$. Then

$$\mathbb{E}g \sum_{i=1}^{n} Y_i = (\mathbb{E}g)\mathbb{E}\left(\sum_{i=1}^{n} Y_i | f = 0\right) \geq n(p - 1/2)\mathbb{E}g$$

because $f = 0$ implies that $\sum_i X_i < n/2$ and $p_i \geq p$. On the other hand,

$$\mathbb{E}g \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} \mathbb{E}Y_i g = \sum_i \mathrm{cov}(X_i, f) = \sum_i p_i(1 - p_i)e_i(f) \leq np(1 - p)\epsilon$$

because the function $q \mapsto q(1 - q)$ is decreasing on the interval $(1/2, 1]$. Combining these two inequalities yields $n(p - 1/2)\mathbb{E}g \leq np(1 - p)\epsilon$, from which we deduce that $\mathbb{E}f \geq 1 - \epsilon p(1 - p)/(p - 1/2)$. Since $p(1 - p) \leq 1/4$, we are finished. □

It turns out that the proof also works if f is a weighted majority function. What's more surprising, perhaps, is that *only* weighted majority functions can aggregate well in this setting.

Theorem 4.3 (Haggstrom, Kalai, Mossel [2]). *If f is monotone and fair and it is not a weighted majority then there exists a probability distribution $\mathbb{P}$ so that $\mathbb{E}X_i > 1/2$ for all i and $\mathbb{E}f = 0$ and $e_i(f, \mathbb{P}) = 0$ for all i.*

References

[1] E. Friedgut and G. Kalai. Every monotone graph property has a sharp threshold. *Proceedings of the American Mathematical Society*, 124(10):2993–3002, 1996.

[2] O. Häggström, G. Kalai, and E. Mossel. A law of large numbers for weighted majority. *Advances in Applied Mathematics*, 37(1):112–123, 2006.

[3] M. Talagrand. On Russo's approximate zero-one law. *The Annals of Probability*, 22(3):1576–1587, 1994.