Social Choice and Networks **Lecture Date: October 5, 2010**

Lecture: Errors in Binary Voting

Elchanan Mossel *Scribe: Nima Anari*

In the next few lectures the focus is going to move away from aggregation of signals. We will instead focus on cases in which voters are unbiased. In this scenario, the focus in no longer on aggregation. Instead the focus shifts to other properties of the systems such as robustness to errors and manipulation.

1 Motivation

Why should we study an unbiased scenario? Well, it can be thought of as a stress-test for the voting methods under consideration. When there is a strong bias, many voting methods are robust to errors, manipulations etc. In order to compare different voting methods it makes sense to stress-test their behavior under the most random scenario, i.e., the uniform measure over voters.

2 Voting Schemes

Let us define the voting systems we are studying.

Definition 1. A *voting scheme* is a function $f : \{-1, 1\}^n \to \{-1, 1\}$.

According to this definition, there are two candidates, namely -1 and 1, and there are n voters casting their direct votes for each candidate. The function f receives a *configuration* of votes and determines the outcome.

As in the previous lectures, we have some natural assumptions about the function f.

Definition 2. A voting scheme is *monotone* if for each $x, y \in \{-1, 1\}^n$ we have $x \geq y \implies f(x) \geq f(y)$. Here $x \geq y$ means that $x_i \geq y_i$ for each i.

The assumption of being monotone is natural. Being monotone means that if some voters switch their votes in favor of the winning candidate, the losing candidate would not suddenly become the winner.

Definition 3. A voting scheme is *fair* if for each $x \in \{-1, 1\}^n$ we have $f(-x) = -f(x)$.

Being fair means being indifferent to names of the candidates. In other words if we replace the names of the two candidates, that is if we switch each -1 by 1 and vice versa, then the same person is still going to win, under her new name.

Example 4. Two important family of voting schemes are MAJORITY and ELECTORAL COLLEGE, both of which are fair and monotone. Assume that n is an odd number, then

- MAJORITY is the function f that associates the most frequent vote as the result. In other words $f(x_1, \dots, x_n) = \text{sign}(x_1 + \cdots + x_n)$.

- Assuming that $n = m^2$, ELECTORAL COLLEGE is a two-level majority vote with m states each holding a population of m. In ELECTORAL COLLEGE we first compute the majority of $x_1, \ldots, x_m$, the majority of $x_{m+1}, \ldots, x_{2m}$ and so on. At the end we end up with m numbers in $\{-1, 1\}$, and we simply compute another majority among these m numbers to determine the final result.

The two assumptions of being monotone and fair are natural enough to assume for every voting scheme under consideration. But there are families of voting schemes that seem unnatural, but still satisfy these two assumptions.

Example 5. The most important example of an unusual voting scheme, is the DICTATOR family. As its name suggests there is a voter i who determines the final outcome. More precisely there is an $i \in [n]$ for which $f(x_1, \ldots, x_n) = x_i$. It's easy to see that this function is both monotone and fair, but in some sense it's unusual because different voters are treated differently.

We are sometimes interested only in the functions which are democratic; functions that do not differentiate between voters.

Definition 6. A voting scheme f satisfies *weak democracy* if there exists a transitive permutation group $\Gamma \leq S_n$ such that for all $\sigma \in \Gamma$ and $x = (x_1, \ldots, x_n) \in \{-1, 1\}^n$ we have

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$$

The transitivity assumption, that for each $i, j \in [n]$ there exists a $\sigma \in \Gamma$ for which $\sigma(i) = j$, ensures that voters i and j have in some sense equal power.

One might ask why we didn't consider all permutations, and instead restricted ourselves to a subgroup.

Definition 7. A voting scheme f satisfies *strong democracy* if for every possible permutation $\sigma \in S_n$ and every voting configuration $x \in \{-1, 1\}^n$ we have

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$$

To answer the question, it's easy to see that strong democracy, along with monotonicity and fairness, implies being MAJORITY. So if we restrict ourselves to strongly democratic functions, we're restricting ourselves to a very special class of functions. On the other hand there are interesting functions that satisfy weak democracy, but not strong democracy. ELECTORAL COLLEGE is one example.

Proposition 8. *A monotone and fair voting scheme that satisfies strong democracy is* MAJORITY.

Proof. Strong democracy implies that f is only a function of the number of 1's among $x_1, \ldots, x_n$. Monotonicity implies that there is a k for which $f(x) = 1$ if and only if there are at least k votes in favor of 1. Now consider two configurations, one with k votes for 1 and the other one with $k - 1$ votes for 1. Using fairness, a configuration with $n - k$ votes for 1 should result in -1 winning and a configuration with $n - k + 1$ votes for 1 should result in 1 winning. Therefore $n - k < k$ and $n - k + 1 \geq k$. Therefore $n - k + 1 = k$ or in other words $k = (n + 1)/2$. This shows that f is MAJORITY. ∎

3 Worst Case vs. Average Case

As mentioned previously, we can not hope for a voting scheme which is error-tolerant in the worst case. The following proposition formalizes this fact.

Proposition 9. *Any non-constant voting scheme is prone to errors of a single voter (single-flip errors).*

Proof. By assumption, there exist configurations x, y for which $f(x) \neq f(y)$. Now consider the configurations $z^i = (x_1, \dots, x_i, y_{i+1}, \dots, y_n)$. We have $z^0 = y$ and $z^n = x$. If there exists an i for which $f(z^i) \neq f(z^{i+1})$, we are done, because z^i and z^{i+1} differ in only one coordinate. But if no such i exists we must have $f(y) = f(z^0) = f(z^1) = \cdots = f(z^n) = f(x)$ which is a contradiction. □

In some sense single-flip errors are the smallest possible set of errors one can consider. So this proposition refutes any hope for a positive result in the worst case scenario.

Now we turn to the average case scenario. We have to define the distribution on the set of possible configurations and the distribution on the set of possible errors that can arise, in order to analyze the average case. The following problem formulates this in a precise manner.

Problem 10. Suppose that $x \in \{-1, 1\}^n$ is a random variable with uniform distribution. In other words $x = (x_1, \dots, x_n)$ where $x_1, \dots, x_n$ are independent random variables each taking values in $\{-1, 1\}$ and having expected value 0. Now let $y = N_\epsilon(x)$ be the random variable obtained from x by flipping each of the coordinates with probability ϵ. That is $y_i = x_i$ with probability ϵ and $y_i = -x_i$ with probability $1 - \epsilon$. Now for a given voting scheme f, what is $P[f(x) \neq f(y)]$?

This probability which we denote by $S_f(\epsilon)$ shows how much the function f tolerates errors. The most stable function has the lowest $S_f(\epsilon)$ and the most sensitive function has the highest $S_f(\epsilon)$.

While the last problem asks to calculate this quantity, perhaps a more interesting question is to find functions f which are stable. Of course constant functions are not appreciated, so f must be a sensible function; maybe a monotone and fair function which perhaps satisfies weak democracy.

4 Stability without Democracy

Suppose that we only require f to be monotone and fair. Then what function(s) would be the most stable?

Theorem 11. DICTATOR *is the unique (up to the choice of dictator) most stable voting scheme.*

Proof. Let N be a $2^n \times 2^n$ matrix where each row and each column corresponds to one element of $\{-1, 1\}^n$. Now let the entry corresponding to row $\hat{x}$ and column $\hat{y}$, namely $N(\hat{x}, \hat{y})$, be equal to the probability of getting $\hat{y}$ after flipping each coordinate of $\hat{x}$ with probability ϵ. It's easy to see that N is symmetric, because each entry only depends on the number of coordinates the vectors corresponding to the row and the column differ on. So $N(\hat{x}, \hat{y}) = N(\hat{y}, \hat{x})$.

It's easy to see that $E[x_i y_i] = P[x_i = y_i] \times 1 + P[x_i \neq y_i] \times -1 = (1 - \epsilon) - \epsilon = 1 - 2\epsilon$ is the covariance between x_i and y_i because both have mean 0. For some reason it's easier to work with $1 - 2\epsilon$ than ϵ, so let $\eta = 1 - 2\epsilon = E[x_i y_i]$.

A function $f : \{-1,1\}^n \to \{-1,1\}$ can be thought of as a vector of dimension 2^n, where each dimension corresponds to a $\hat{x} \in \{-1,1\}^n$. Because f is fair we have

$$E[f(x)] = E[-f(-x)] = -E[f(-x)]$$

but $-x$ has the same distribution as x, which shows that $E[f(x)] = 0$. Similarly $E[f(y)] = 0$, because y has the same distribution as x. Now note that

$$E[f(x)f(y)] = P[f(x) = f(y)] \times 1 + P[f(x) \neq f(y)] \times -1 = (1 - S_f(\epsilon)) - S_f(\epsilon) = 1 - 2S_f(\epsilon)$$

So instead of $S_f(\epsilon)$ we can study the quantity $Z(f,\eta) = E[f(x)f(y)] = 1 - 2S_f(\epsilon)$ and try to maximize it (we were after minimizing $S_f(\epsilon)$). But note that $E[f(x)f(y)] = E[E[f(x)f(y)|x]] = E[f(x)E[f(y)|x]]$. And we can calculate $E[f(y)|x]$ using N. We have

$$E[f(y)|x] = \sum_{\hat{y} \in \{-1,1\}^n} P[y = \hat{y}|x]f(\hat{y}) = \sum_{\hat{y}} N(x,\hat{y})f(\hat{y})$$

which when represented as a vector (indexed by x) is equal to Nf. Therefore we have

$$Z(f,\eta) = E[f(x) \times (Nf)(x)]$$

which reminds us of the Fourier analysis. The tools we need to use from Fourier analysis can be found in the appendix. Here we assume sufficient familiarity with Fourier analysis of Boolean functions. Let $\{u_S | S \subseteq [n]\}$ be the usual Fourier basis. Also let $< f, g >= E[fg]$ be the usual inner product defined on the space of functions (or vectors of dimension 2^n). We have $Z(f,\eta) =< f, Nf >$. Now let's write f in the Fourier basis:

$$f = \sum_{S \subseteq [n]} f_S u_S$$

One magical property of u_S is that it's an eigenvector of N. In order to show this, we have to show that Nu_S is a multiple of u_S. But we already know that Nu_S can be written as a linear combination of u_T's. So we have to show that each coefficient in this linear combination is 0 except for u_S's coefficient. Because of the orthonormality of the basis, the coefficient of u_T is equal to

$$< u_T, Nu_S >= E[u_T(x) \times (Nu_S)(x)] = E[u_T(x)E[u_S(y)|x](x)] = E[u_T(x)u_S(y)]$$

This equalities follow in the same way as we have shown $< f, Nf >= E[f(x)f(y)]$. Now note that

$$\begin{aligned} E[u_T(x)u_S(y)] &= E[\prod_{i \in T} x_i \prod_{i \in S} y_i] = E[\prod_{i \in S \cap T} x_i y_i \prod_{i \in S \setminus T} y_i \prod_{i \in T \setminus S} x_i] \\ &= \prod_{i \in S \cap T} E[x_i y_i] \prod_{i \in S \setminus T} E[y_i] \prod_{i \in T \setminus S} E[x_i] \end{aligned}$$

where the last equality follows because of the independence of each pair of x_i, y_i from all other random variables. We know that $E[x_i] = E[y_i] = 0$ and $E[x_i y_i] = \eta$. Therefore if either $S \setminus T$ or $T \setminus S$ are non-empty the above expression is equal to 0. It's easy to see that for $S \neq T$ one of these two sets is non-empty. When $S = T$ the expression above is equal to $\eta^{|S|}$.

Therefore the Fourier coefficient of Nu_S with respect to u_T is equal to 0 when $S \neq T$ and is equal to $\eta^{|S|}$ when $S = T$. So $Nu_S = \eta^{|S|} u_S$. Now back to the original problem, we have

$$< f, Nf >=< \sum_S f_S u_S, \sum_S \eta^{|S|} f_S u_S >= \sum_S \eta^{|S|} f_S^2$$

Note that for $S = \emptyset$ we have $f_\emptyset =< f, 1 >= E[f] = 0$. So the above sum can be written as

$$\sum_{S \neq \emptyset} \eta^{|S|} f_S^2 = \eta \sum_{S \neq \emptyset} \eta^{|S|-1} f_S^2$$

Now note that $|\eta^{|S|-1}| \leq 1$ because $|\eta| = |1 - 2\epsilon| \leq 1$. So an upper-bound for $Z(f, \eta)$ would be

$$Z(f, \eta) \leq \eta \sum_{S \neq \emptyset} f_S^2 = \eta < \sum_S f_S u_S, \sum_S f_S u_S >= \eta < f, f >$$

But note that $< f, f >= E[f^2] = E[1] = 1$ because f takes values in $\{-1, 1\}$. So

$$Z(f, \eta) \leq \eta$$

When does equality happen? It's easy to see that equality happens if and only if for each $S \neq \emptyset$ we have $\eta^{|S|-1} f_S^2 = f_S^2$. This can only happen when $|S| = 1$ or $f_S = 0$ (assuming that $\eta \neq 0$). So all the Fourier coefficients of f except for those corresponding to singletons are zero. So

$$f(x) = \sum_{i \in [n]} f_{\{i\}} u_{\{i\}}(x) = \sum_{i \in [n]} f_{\{i\}} x_i$$

Note that $f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, -1, x_{i+1}, \ldots, x_n) = 2f_{\{i\}}$. But since f is monotone and takes values in $\{-1, 1\}$ this quantity must be inside $\{0, 2\}$. So $2f_{\{i\}} \in \{0, 2\}$ which means that $f_{\{i\}} \in \{0, 1\}$. But again we know that $\sum_i f_{\{i\}}^2 = 1$, so there must be a unique index i for which $f_{\{i\}}$ is nonzero. Therefore f is equal to x_i for some i. □

Remark 12. The above proof fails when $\eta = 0$, or equivalently when $\epsilon = 1/2$. But this is no surprise, because when $\epsilon = 1/2$, essentially y is independent of x. So the probability of $f(x) \neq f(y)$ is exactly 1/2 for any fair function, because half of the configurations result in 1 and the other half result in -1.

5 Stability with Democracy

Now let's find out how stable can a democratic function be. One natural function to examine is MAJORITY. The following theorem states the stability of MAJORITY when the number of voters goes to infinity.

Theorem 13. *(Sheppard [4])*[1] *If f is the* MAJORITY *in a population of n, then*

$$\lim_{n \to \infty} Z(f, \eta) = \frac{2 \arcsin(\eta)}{\pi}$$

where Z and η are the same notations from the proof of theorem 11.

Proof. The result can be obtained after a standard application of the two-dimensional central limit theorem. Let $z_i = (x_i, y_i)$ be the two-dimensional vector. Clearly $z_1, z_2, \ldots$ are independent and identically distributed. It's easy to see that $E[z_i] = (0, 0)$. The covariance matrix for each vector is equal to

$$\Sigma = \begin{bmatrix} E[x_i^2] & E[x_i y_i] \\ E[x_i y_i] & E[y_i^2] \end{bmatrix} = \begin{bmatrix} 1 & \eta \\ \eta & 1 \end{bmatrix}$$

[1] In the slides, the theorem is said to be due to Sheffield, but I believe the correct person is Sheppard.

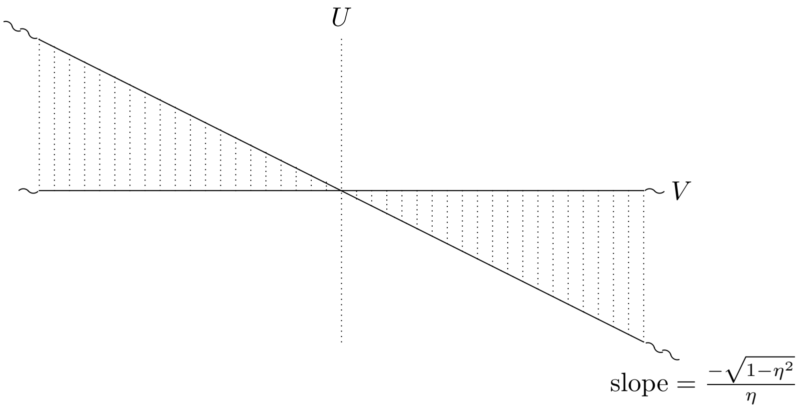

Figure 1: The event $\mathrm{sign}(\sqrt{1-\eta^2}V + \eta U) \neq \mathrm{sign}(U)$

Therefore the two-dimensional central limit theorem states that

$$\frac{z_1 + \cdots + z_n}{\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, \Sigma)$$

where $\xrightarrow{D}$ means convergence in distribution. Note that $f(x_1, \ldots, x_n) = \mathrm{sign}(x_1 + \cdots + x_n) = \mathrm{sign}(\frac{x_1+\cdots+x_n}{\sqrt{n}})$. Now let $N_n = \frac{x_1+\cdots+x_n}{\sqrt{n}}$ and $M_n = \frac{y_1+\cdots+y_n}{\sqrt{n}}$. We're interested in $Z(f, \eta) = E[\mathrm{sign}(N_n)\mathrm{sign}(M_n)]$. But we know that $(N_n, M_n) \xrightarrow{D} (N, M)$ where (N, M) is a multivariate Gaussian $\mathcal{N}(0, \Sigma)$. So the limit of the probability is equal to

$$E[\mathrm{sign}(N)\mathrm{sign}(M)]$$

Here we can write a complicated integral and calculate the desired value. But instead, we use a simple fact about multi-variate Gaussians: if U, V are random variables in $\mathrm{span}\{N, M\}$ for which $\mathrm{Cov}(U, V) = 0$, then U, V are independent. Now let $U = N$ and $V = aN + bM$. Then

$$\begin{aligned}\mathrm{Cov}(U, V) &= a\mathrm{Var}(N) + b\mathrm{Cov}(N, M) = a + b\eta \\ \mathrm{Var}(V) &= a^2\mathrm{Var}(N) + b^2\mathrm{Var}(M) + 2ab\mathrm{Cov}(N, M) = a^2 + b^2 + 2ab\eta\end{aligned}$$

We want the covariance matrix of U, V to be equal to identity. So we must have $a + b\eta = 0$ and $a^2 + b^2 + 2ab\eta = 1$. One choice is $a = -\eta/\sqrt{1-\eta^2}$ and $b = 1/\sqrt{1-\eta^2}$. One can easily see that $M = (V - aU)/b = \sqrt{1-\eta^2}V + \eta U$. So we want to find out

$$E[\mathrm{sign}(\sqrt{1-\eta^2}V + \eta U)\mathrm{sign}(U)] = 1 - 2P[\mathrm{sign}(\sqrt{1-\eta^2}V + \eta U) \neq \mathrm{sign}(U)]$$

where U, V are independent $\mathcal{N}(0, 1)$ random variables. Now take a look at the joint probability distribution of (U, V) in the plane. The hashed region in figure 1 shows the region corresponding to the event $\mathrm{sign}(N) \neq \mathrm{sign}(M)$.

So to compute the probability, we have to integrate the normal distribution over the hashed region. There are several ways to do this, but the easiest one is to integrate in the polar coordinates. If Δ denotes the hashed region, and $\phi(u, v)$ denoted the p.d.f. for the normal distribution, then

$$\int\int_{\Delta} \phi(u, v)\,du\,dv = \int_r \int_{0 \geq \tan\theta \geq -\sqrt{1-\eta^2}/\eta} \phi(r\cos\theta, r\sin\theta)\,d\theta \times 2\pi r\,dr$$

Now if we look at the inner integral, it's just an integral over a constant function, because ϕ is invariant under rotations. So the integral is proportional to the length of the integration interval

Value of ϵ	Error probability in MAJORITY	Error probability in DICTATOR
1%	$10\% \times \frac{2}{\pi} \simeq 6.4\%$	1%
0.01%	$1\% \times \frac{2}{\pi} \simeq 0.6\%$	0.01%

Table 1: Errors in MAJORITY and DICTATOR

(actually, there are two intervals). So if we compare this to the integral over the whole circle it's going to be exactly $2\arctan(\sqrt{1-\eta^2}/\eta)/2\pi$ times smaller. Therefore

$$\int\int_{\Delta} \phi(u,v)dudv = \frac{2\arctan(\sqrt{1-\eta^2}/\eta)}{2\pi} \int_r \int_{2\pi \geq \theta \geq 0} \phi(r\cos\theta, r\sin\theta)d\theta \times 2\pi r dr$$

But the integral on the right hand side is just the integral over the whole plane of the normal distribution, which is equal to 1. So

$$P[\text{sign}(\sqrt{1-\eta^2}V + \eta U) \neq \text{sign}(U)] = \frac{\arctan(\sqrt{1-\eta^2}/\eta)}{\pi}$$

Now the limit of $Z(f,\eta)$ is equal to

$$\lim_{n\to\infty} Z(f,\eta) = 1 - 2\frac{\arctan(\sqrt{1-\eta^2}/\eta)}{\pi} = 1 - 2\frac{\arccos\eta}{\pi} = \frac{2\arcsin\eta}{\pi}$$

□

In order to estimate $S_f(\epsilon)$ when ϵ is small and n is large, note that

$$S_f(\epsilon) = \frac{1 - Z(f,\eta)}{2} \simeq \frac{\pi - 2\arcsin\eta}{2\pi} = \frac{\arccos\eta}{\pi}$$

In particular note that when $\epsilon \to 0$, we have $\eta \to 1$ and so $S_f(\epsilon) \to 0$. So

$$\epsilon = \frac{1-\eta}{2} \simeq \frac{1 - \cos(\pi S_f(\epsilon))}{2} \simeq \frac{\pi^2 S_f(\epsilon)^2}{4}$$

where we used the first nonzero Taylor expansion term in the last step. This means that

$$S_f(\epsilon) \simeq \frac{2}{\pi}\sqrt{\epsilon}$$

In comparison, for DICTATOR the result was

$$S_f(\epsilon) = \epsilon$$

which is far better than MAJORITY. To get a better sense of the difference between these two errors, look at table 1.

Example 14. Using the estimation for MAJORITY, one can also estimate the probability of error in the ELECTORAL COLLEGE. Given that the probability of error for an individual is ϵ, the probability of error for a state is roughly $2\sqrt{\epsilon}/\pi$. Now we can treat each state as an individual and use the estimation one more time to see that the probability of error for the entire election is equal to $2\sqrt{2\sqrt{\epsilon}/\pi}/\pi = \Theta(\sqrt[4]{\epsilon})$, which is worse than MAJORITY.

The natural question to ask is whether we can come up with a democratic function that is more stable than MAJORITY. The answer is no; the following theorem states that MAJORITY is essentially the best among the set of functions with low influence (which includes the class of weakly democratic functions).

Theorem 15. *(Mossel, et al. [3]) If a family of functions $f_n : \{-1,1\}^n \to \{-1,1\}$ satisfy $E[f_n] = 0$ and*

$$\max_{1\le i\le n} \{Inf_i(f_n)\} = o(1)$$

where $Inf_i(f_n) = E_x[Var_{x_i}[f(x)]]$ is the influence of the i-th voter, then

$$\lim_{n\to\infty} S_{f_n}(\epsilon) \ge \frac{1}{2} - \frac{\arcsin(1-2\epsilon)}{\pi}$$

The proof is not going to presented here, but note that the bound proven is tight for MAJORITY, which means that asymptotically MAJORITY is the best we can get.

Remark 16. It can be shown that a family of weakly democratic function satisfies the conditions of theorem 15. Because $\mathrm{Inf}_i(f_n)$ is the same for all i and so $\mathrm{Inf}_i(f_n) = \frac{1}{n}E_x[\sum_j \mathrm{Var}_{x_j}[f(x)]]$. As we will see in the next section this quantity is maximized for MAJORITY. For MAJORITY it's roughly equal to $\theta(\frac{1}{\sqrt{n}}) = o(1)$.

6 A Different Error Model

Interestingly, in another reasonable model of errors, MAJORITY is the worst function. In this model the vector y of registered votes is obtained from x not by flipping each coordinate randomly, but by flipping exactly one random coordinate. So $y = (x_1, \ldots, -x_i, \ldots, x_n)$ with probability $1/n$ for $i = 1, \ldots, n$.

Proposition 17. *Under the single coordinate-flip error model,* MAJORITY *has the most probability of error among all monotone functions.*

Proof. Consider all possible configurations for (x, y). There are exactly $n2^n$ such configurations, all happening with equal probabilities. Let the set of all possible configurations be $S \subseteq \{-1,1\}^{2n}$. Then we want to find the function f which maximizes

$$\sum_{(x,y)\in S} |f(x) - f(y)|$$

Now let's partition $S = S_1 \cup S_2$, where S_1 is the set of (x, y) for which $x > y$ and S_2 is the set of (x, y) for which $x < y$. Because x, y differ in only one position, one of $x < y$ and $y < x$ must be true for them, so this is a valid partitioning. Now because f is monotone we have

$$\sum_{(x,y)\in S} |f(x) - f(y)| = \sum_{(x,y)\in S_1} (f(x) - f(y)) + \sum_{(x,y)\in S_2} (f(y) - f(x))$$

Given some fixed x, the number of y's for which $(x, y) \in S_1$ is equal to $\#_1(x)$ where $\#_s(x)$ denotes the number of coordinates of x which are equal to s. Similarly the number of y's for which $(x, y) \in S_2$ is equal to $\#_0(x)$. We get similar numbers when we fix y. So in the end, the above sum can be written as

$$\sum_x (\#_1(x) - \#_0(x))f(x) + \sum_y (\#_1(y) - \#_0(y))f(y) = 2\sum_x (\#_1(x) - \#_0(x))f(x)$$

Now putting aside the monotonicity condition, it's clear that in order to maximize the above quantity we have to set $f(x) = \text{sign}(\#_1(x) - \#_0(x))$ which is exactly the same as MAJORITY, which happens to be a monotone function too. □

As we have speculated at the end of the last section, the probability of error in the new model has something to do with the quantity $I(f) := E_x[\sum_i \text{Var}_{x_i}[f]]$. In fact it's relatively easy to see that this probability is exactly equal to $\frac{1}{n} E_x[\sum_i \text{Var}_{x_i}[f]]$.

For MAJORITY we have $I(f)^2 \simeq 2n/\pi$, because there are approximately $n\binom{n}{n/2}$ pairs of adjacent x, y for which MAJORITY gives different answers; so the probability of error is roughly equal to $n\binom{n}{n/2}/(n2^n)$ which is approximately $\sqrt{2/(\pi n)}$. So $I(f)$ which is n times this probability is roughly equal to $\sqrt{2n/\pi}$. Note that as n goes to infinity, the error probability gets arbitrarily close to zero.

The relationship between the new error model and the previous error model is not obvious at first, but it can be seen that in the previous model when ϵ is no longer a constant and satisfies $\epsilon \ll 1/n$, we approximately get the new model. Because when $\epsilon \ll 1/n$ the probability of having at least two flips is much smaller than the probability of having one flip.

In sum, when $\epsilon = \Omega(1)$, MAJORITY is the most stable and when $\epsilon = o(1/n)$, MAJORITY is the most sensitive.

Even though MAJORITY is the most sensitive function in the new error model, the probability that it reports an incorrect outcome is still not as large as a constant. A natural extension of this model is to flip not just 1 but $g(n)$ random coordinates where $g(n)$ is a function of n. How large should $g(n)$ be in order to get a constant error probability? This question is interesting in many contexts like learning, neural networks, hardness amplification and etc.

Again we fall back on our previous model, but with a non-constant flip probability. The question is how large should ϵ be to get a constant $S_f(\epsilon)$. We are still interested in fair monotone functions.

Using the Fourier analysis we can express a function f as $\sum_{S \subseteq [n]} f_S u_S$. The previous proposition shows that $I(f)^2 \preceq 2n/\pi$ where $\preceq$ means being smaller asymptotically. But note that $\text{Inf}_i(f) = \sum_{i \in S} f_S^2$, so

$$I(f) = \sum_i \text{Inf}_i(f) = \sum_S |S| f_S^2$$

This can also be shown using the Russo's formula from lecture 1. As we have seen in the proof of theorem 11 the error probability when we flip each coordinate with probability ϵ is equal to

$$\frac{1 - \sum_S \eta^{|S|} f_S^2}{2}$$

where $\eta = 1 - 2\epsilon$. So in order to maximize this quantity, we have to minimize

$$\sum_S \eta^{|S|} f_S^2$$

Let $x_S = f_S^2$. Now relaxing the condition of f being a fair monotone function whose range is $\{-1, 1\}$, we get the following optimization problem

$$\begin{aligned} \text{Minimize} \quad & \sum_S \eta^{|S|} x_S \\ \text{Subject to} \quad & \sum_S |S| x_S \leq \alpha \simeq \sqrt{2\pi/n} \\ & \sum_S x_S = 1 \\ & x_S \geq 0 \end{aligned}$$

The equality $\sum_S x_S = 1$ is true before relaxation, because $\sum_S f_S^2 =< f, f >= 1$. Now because η^x is a convex function, using the Jensen's inequality we get

$$\sum x_S \eta^{|S|} \geq \eta^{\sum_S x_S |S|} \geq \eta^\alpha$$

Therefore this bound is also true for any function f. So $Z(f, \eta) \geq \eta^\alpha$ where $\alpha \simeq \sqrt{2n/\pi}$. We want $Z(f, \eta)$ to be smaller than a constant less than 1, in order to get constant error probability. So η^α has to be smaller than a constant less than 1. But we have

$$\eta^\alpha = (1 - 2\epsilon)^\alpha \geq 1 - 2\alpha\epsilon$$

So $2\alpha\epsilon$ has to be $\Omega(1)$ which means that $\epsilon = \Omega(1/\sqrt{n})$. So we have to flip approximately $\theta(\sqrt{n})$ coordinates to get a constant error probability. Note that because of the concentration bounds it makes sense to speak of flipping each coordinate with probability $1/\sqrt{n}$ and flipping $\sqrt{n}$ random coordinates, interchangeably.

We have shown a lower bound, but is it tight? This question was first asked by (Kalai, et al. [1]). The following theorem shows that the bound is roughly tight. They consider a family of functions REC-MAJORITY-k where we do a recursive majority vote. At each step we do a majority among groups of k voters to reduce the number of votes by a factor of k and we repeat this until we reach the final vote.

Theorem 18. *(Mossel, et al. [2]) The family of functions* REC-MAJORITY-k *satisfies*

$$Z(f, \eta) \preceq \eta^{\alpha(n,k)}$$

where $\alpha(n, k) \simeq n^{\beta(k)}$ *for some function* β *which satisfies* $\lim_{k\to\infty} \beta(k) = 1/2$.

The above theorem shows that it's enough to roughly flip $n^{1-\beta(k)}$ votes to get a constant error probability. So we can come as close as we want in the exponent to 1/2 which is essentially our lowerbound.

Finally a slight modification of a randomized construction by (Talagrand [5]) reaches the lowerbound. That is there is a family of randomly constructed functions for which flipping $c\sqrt{n}$ votes would result in a constant error probability. For the analysis see (Mossel, et al. [2]). But note that this family is not explicitly constructed and so the previous result about recursive majorities is not entirely weaker than this result.

7 Final Remarks

Up until now we have been studying the scenario in which we're comparing two possible outcomes, the true original outcome and the outcome after noise. One might consider comparing more than two outcomes. Suppose that there is a vector x of original votes, and through k independent experiments we get k vectors $y_1, \ldots, y_k$. We can assume that each y_i is obtained from x by randomly flipping each vote with probability ϵ. Then the stability of our function f can be measured by

$$P[f(y_1) = \cdots = f(y_k) = f(x)]$$

The equality with $f(x)$ is optional; if we omit it, little is going to change. The intuition is that the probability of error is small, so the probability of having k errors is going to be really small compared to the probability of having no errors.

Now when $k \leq 3$ it can be shown that DICTATOR is the most stable function. On the other hand if n is fixed and $k \to \infty$, then MAJORITY is going to be the most stable function. Strangely, if k is something in between, then neither MAJORITY nor DICTATOR is going to be the most stable function. Taking the majority over a subset of voters would result in a better stability than both MAJORITY and DICTATOR (note that we get MAJORITY when this subset is the whole set and we get DICTATOR when the subset is a singleton).

References

[1] G. Kalai, J. Kahn, and N. Linial. The influence of variables on boolean functions. *Proceedings of the 29th Annual Symposium on Foundations of Computer Science*, pages 68–80, 1988.

[2] E. Mossel and R. O'Donnell. On the noise sensitivity of monotone functions. *Random Structures and Algorithms*, 23(3):333–350, 2003.

[3] E. Mossel, R. O'Donnell, and K. Oleszkiewicz. Noise stability of functions with low influences: invariance and optimality. *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science*, pages 21–30, 2005.

[4] W. F. Sheppard. On the application of the theory of errors to cases of normal distribution and normal correlation. *Philosophical Transactions of the Royal Society of London*, 192:101–167, 1899.

[5] M. Talagrand. How much are increasing sets positively correlated? *Combinatorica*, 16(2):243–258, 1996.

Appendix: Fourier Analysis

The space of all functions $f : \{-1,1\}^n \to \mathbb{R}$ is a linear space of dimension 2^n. We can think of functions as vectors with 2^n coordinates, each corresponding to an element in $\{-1,1\}^n$. We can equip our space with an inner product

$$< f, g >= E[fg] = \frac{1}{2^n} \sum_{x \in \{-1,1\}^n} f(x)g(x)$$

Now let S be a subset of $[n] = \{1, \ldots, n\}$. Let $u_S : \{-1,1\}^n \to \mathbb{R}$ be the function defined by

$$u_S(x) = \prod_{i \in S} x_i$$

One property of these functions is that $u_S u_T = u_{S\Delta T}$ where $S\Delta T = (S \backslash T) \cup (T \backslash S)$. This is easy to see because $x_i^2 = x_i^0 = 1$. So the terms x_i which appear in both u_S and u_T vanish in the product. Now using this property we know that

$$< u_S, u_T >= E[u_S u_T] = E[u_{S\Delta T}] = E[\prod_{i \in S\Delta T} x_i] = \prod_{i \in S\Delta T} E[x_i]$$

Now if $S\Delta T$ is nonempty, the above product is 0, because $E[x_i] = 0$. If $S\Delta T = \emptyset$ then the above product is 1. So we have

$$< u_S, u_T >= \begin{cases} 1 & S = T \\ 0 & S \neq T \end{cases}$$

The set $\{u_S | S \subseteq [n]\}$ consists of 2^n elements, which is equal to the dimension of the space. The functions are orthonormal as we have shown, so they are linearly independent and form a basis for the space. So each function f can be written as $f(x) = \sum_S f_S u_S(x)$. Note that

$$< f, u_S >=< \sum_T f_T u_T, u_S >= f_S$$

so we have an easy way to express f_S.

The inner product $< f, g >$ of two functions can be written as

$$< f, g >=< \sum_S f_S u_S, \sum_T g_T u_T >= \sum_{S,T} f_S g_T < u_S, u_T >= \sum_S f_S g_S$$

Particularly $< f, f >= E[f^2] = \sum_S f_S^2$. So if $f^2 = 1$ or in other words $\forall x : f(x) \in \{-1, 1\}$, we have $\sum_S f_S^2 = 1$.

One of the interesting applications of Fourier analysis is to express $\text{Inf}_i(f)$. We will show that

$$\text{Inf}_i(f) = \sum_{S \ni i} f_S^2$$

This is true because if we consider the function $g(x) = \frac{1}{2}[f(x) - f(x_1, \ldots, -x_i, \ldots, x_n)]$, then

$$g(x)^2 = \text{Var}_{x_i}[f]$$

because f takes values in $\{-1, 1\}$ and so the Var_{x_i} is either 0 or 1. $g(x)$ is in $\{-1, 0, 1\}$ and it's 0 exactly when Var_{x_i} is 0. Now note that

$$f(x_1, \ldots, x_n) = \sum_S f_S u_S(x)$$

so

$$f(x_1, \ldots, -x_i, \ldots, x_n) = \sum_S f_S u_S(x_1, \ldots, -x_i, \ldots, x_n)$$

But it's easy to see that $u_S(x_1, \ldots, -x_i, \ldots, x_n)$ equals $u_S(x)$ when $i \notin S$ and equals $u_S(x)$ when $i \in S$. So

$$f(x_1, \ldots, -x_i, \ldots, x_n) = \sum_{S \not\ni i} f_S u_S(x) - \sum_{S \ni i} f_S u_S(x)$$

Now it's easy to see that

$$g(x) = \frac{1}{2}\left(\left(\sum_{S \not\ni i} f_S u_S(x) + \sum_{S \ni i} f_S u_S(x)\right) - \left(\sum_{S \not\ni i} f_S u_S(x) - \sum_{S \not\ni i} f_S u_S(x)\right)\right) = \sum_{S \ni i} f_S u_S(x)$$

Therefore

$$\text{Inf}_i(f) = E[\text{Var}_{x_i}[f]] = E[g^2] =< g, g >= \sum_S g_S^2 = \sum_{S \ni i} f_S^2$$

These are the tools we needed in the previous sections, but there is certainly more to Fourier analysis than these facts.