Arrow Theorem

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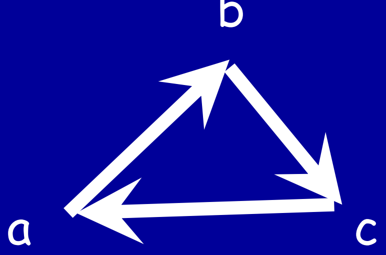
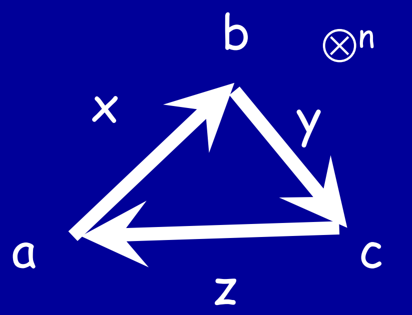
Condorcet Paradox

- n voters are to choose between 3 alternatives.
- Condorcet: Is there a rational way to do it?
- More specifically, for majority vote:
- Could it be that all of the following hold:
  - Majority of voters rank a above b?
  - Majority of voters rank b above c?
  - Majority of voters rank c above a?
- Condorcet(1785): Could be.

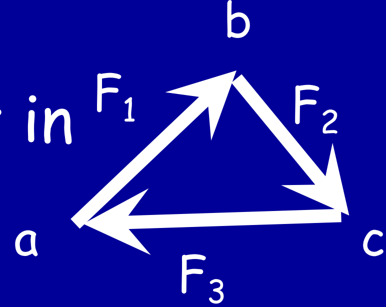


- Defined by Marquis de Condorcet as part of a discussion of the best way to elect candidates to the French academy of Science.

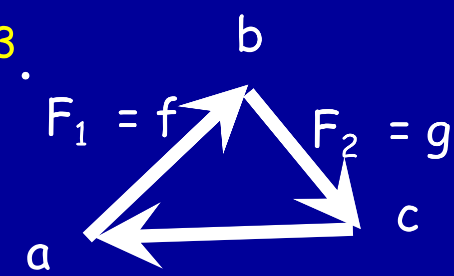
Properties of Constitutions



- n voters are to choose between 3 alternatives
- Voter i ranking := $\sigma_i \in S(3)$. Let:
- $x_i = +1$ if $\sigma_i(a) > \sigma_i(b)$, $x_i = -1$ if $\sigma_i(a) < \sigma_i(b)$,
- $y_i = +1$ if $\sigma_i(b) > \sigma_i(c)$, $y_i = -1$ if $\sigma_i(b) < \sigma_i(c)$,
- $z_i = +1$ if $\sigma_i(c) > \sigma_i(a)$, $z_i = -1$ if $\sigma_i(c) < \sigma_i(a)$.
- Note: (x_i, y_i, z_i) correspond to a σ_i iff (x_i, y_i, z_i) not in $\{(1,1,1),(-1,-1,-1)\}$



- Def: A constitution is a map $F : S(3)^n \to \{-1,1\}^3$.
- Def: A constitution is transitive if for all σ:
  - $F(\sigma) \in \{-1,1\}^3 \setminus \{(1,1,1),(-1,-1,-1)\}$



- Def: Independence of Irrelevant Alternatives (IIA) is satisfied by F if: $F(\sigma) = (f(x), g(y), h(z))$ for all σ and some f, g and h.

Arrow's Impossibility Thm

- Def: A constitution F satisfies Unanimity if $\sigma_1 = \ldots = \sigma_n \Rightarrow F(\sigma_1,\ldots,\sigma_{\underline{n}}) = \sigma_1$
- Thm (Arrow's "Impossibility", 61): Any constitution F on 3 (or more) alternatives which satisfies
- IIA,
- Transitivity and
- Unanimity:

Is a dictator: There exists an i such that:

$F(\sigma) = F(\sigma_1,\ldots,\sigma_n) = \sigma_i$ for all σ

Arrow received a nobel Prize in Economics in 1972

Arrow's Impossibility Thm

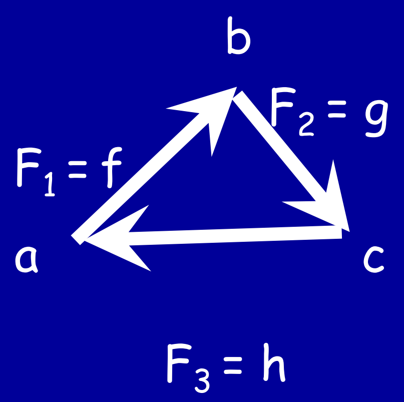
The Royal Swedish Academy of Sciences has
decided to award the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel,
1972 , to

John R Hicks, Oxford University, U K
and
Kenneth Arrow, Harvard University, USA

for their pioneering contributions to general economic equilibrium theory and welfare theory.

A Short Proof of Arrow Thm

- Def: Voter 1 is pivotal for f (denoted $I_1(f) > 0$) if: $f(-,x_2,\ldots,x_n) \neq f(+,x_2,\ldots,x_n)$ for some $x_2,\ldots,x_n$ (similarly for other voters).
- Lemma (Barbera 82): Any constitution $F=(f,g,h)$ on 3 alternatives which satisfies IIA and has
- $I_1(f) > 0$ and $I_2(g) > 0$
- has a non-transitive outcome.



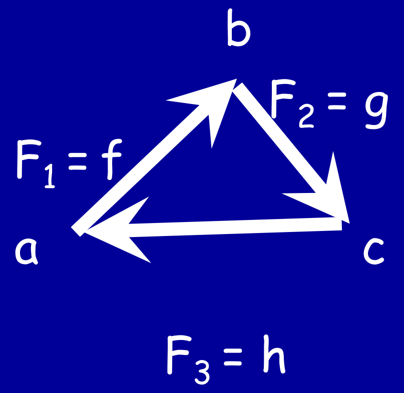
- Pf: $\exists\ x_2,\ldots,x_n$ and $y_1,y_3,\ldots,y_n$ s.t:
- $f(+1,+x_2,+x_3\ldots,+x_n) \neq f(-1,+x_2,+x_3\ldots,+x_n)$
- $g(+y_1,+1,+y_3,\ldots,+y_n) \neq g(+y_1,-1,+y_3,\ldots,+y_n)$
- $h(-y_1,-x_2,-x_3,\ldots,-x_n) := v$ and choose x_1,y_2 s.t.: $f(x) = g(y) = v$ $\Rightarrow$ outcome is not transtive.
- Note: $(x_1,y_1,-y_1),(x_2,y_2,-x_2),(x_i,y_i,-x_i)$ not in $\{(1,1,1),(-1,-1,-1)\}$

A Short Proof of Arrow Thm

- Pf of Arrow Thm:
- Let $F = (f,g,h)$.
- Let $I(f)$ = {pivotal voters for f}.
- Unanimity $\Rightarrow$ f,g,h are not constant

  $\Rightarrow$ $I(f), I(g), I(h)$ are non-empty.
- By Transitivity + lemma $\Rightarrow$ $I(f) = I(g) = I(h) = \{i\}$ for some i.
- $\Rightarrow F(\sigma) = G(\sigma_i)$
- By unanimity $\Rightarrow F(\sigma) = \sigma_i$.

- Q: How to prove for k> 3 alternatives?
- Q: Can we do without unanimity?



A Short Proof of Arrow Thm

- Q: How to prove for k> 3 alternatives?
- A: For each 3 alternatives there is a dictator so we only need to show it is the same dictator for all pairs of alternatives. If {a,b},{c,d} are two such pairs look at (a,b,c) and (b,c,d).

- Q: Can we do without unanimity?
- A: Except the last step the same proof works if instead of unanimity we have that: for each pair of alternatives b in some outcome a beats b and in another b beats a.
- Then in the last step we get $F = G(\sigma_i)$
- Only such F that satisfy IIA is $F(\sigma) = \sigma$ and $F(\sigma) = -\sigma$.

A more general Arrow Theorem

- Def: Write $A >_F B$ if for all σ and all $a \in A$ and $b \in B$ it holds that $F(\sigma)$ ranks a above b.
- Thm (Wilson 72 as stated in M'10): A constitution F on k alternatives satisfies IIA and Transitivity iff
- F satisfies that there exists a partition of the k alternatives into sets $A_1,\ldots A_s$ s.t:
- $A_1 >_F \ldots >_F A_s$ and
- If $|A_r| > 2$ then F restricted to A_r is a dictator on some voter j.

- Note: "Dictator" now is also $F(\sigma) = -\sigma$.
- Def: Let $F_k(n)$:= The set of constitutions on n voters and k alternatives satisfying IIA and Transitivity.

Pf of Wilson's Theorem

- Def: Write $A >_F B$ if for all σ and all $a \in A$ and $b \in B$ it holds that $F(\sigma)$ ranks a above b.
- Thm (Wilson 72 as stated in M'10): A constitution F on k alternatives satisfies IIA and Transitivity iff
- F satisfies that there exists a partition of the k alternatives into sets $A_1,\ldots A_s$ s.t:
- $A_1 >_F \ldots >_F A_s$ and
- If $|A_r| > 2$ then F restricted to A_r is a dictator on some voter j.

- Note: every function as above is IIA and transitive, so need to show that if f is IIA and transitive then satisfies the conditions above.

Pf of Wilson's Theorem

- Assume F is transitive and IIA.
- For two alternatives a,b write $a >_F b$ is a is always ranked above b. Write $a \sim_F b$ if there are outcome where a>b and outcome where b>a.
- <u>Claim:</u> $>_F$ is transitive.
- <u>Claim:</u> If there exists a profile σ where a>b and a profile τ where b>c then there exists an outcome where a>c.
- <u>Pf:</u> As in Barbera pf look at the configuration with a,b preferences taken from σ and b,c preferences taken from τ.
- <u>Claim:</u> $\sim_F$ is transitive moreover if $a >_F b$ and $a \sim_F c$ and $b \sim_F d$ then $c >_F d$.

Pf of Wilson's Theorem

- Claim: $>_F$ is transitive.
- Claim: $\sim_F$ is transitive moreover if $a >_F b$ and $a \sim_F c$ and $b \sim_F d$ then $c >_F d$
- Claim: There exists a partition of the alternatives $A_1 >_F A_2 >_F \ldots >_F A_s$
- Pf of Wilson's theorem: Apply Arrow thm to each of th$_i$A_i's.

Ties

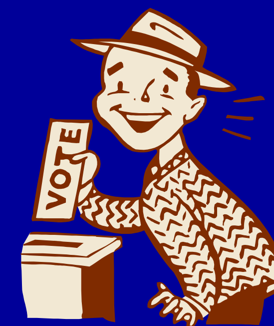
- Note: So far we assumed that each voters provides a strict ranking.
- Arrow and other work considered the more general case where voters are allowed to have a ranking with ties such as:
- a > b~c or a~b > c etc.
- Under this condition one can state Arrow's and Wilson's theorems but only one sided versions:
- Arrow theorem with ties:
- If F satisfies unanimity, IIA and transitivity then it is a dictator or null where
- Def: Dictator is a voter whose strict preferences are followed.

Some Examples of dictators

- Example 1: $F(\sigma) = \sigma_1$.
- Example 2: All the strict inequalities of σ_1 are followed and:
- for every pair of alternatives a~b in σ_1 run a majority vote on the pairwise preferences between a and b.
- Note:
- Example 1 satisfies IIA while example 2 doesn't.
- If and only if characterization in M-Tamuz-11.

Random Ranking:

- Assume uniform voting

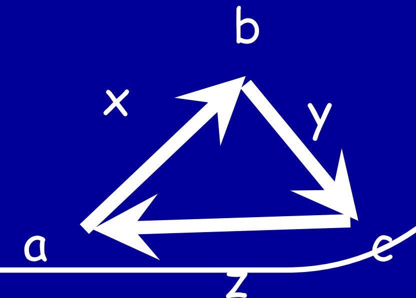


- Note: Rankings are chosen uniformly in S_3^n
- Assume IIA: $F(\sigma) = (f(x), g(y), h(z))$
- Q: What is the probability of a paradox:
- Def: $PDX(F) = P[f(x) = g(y) = h(z)]$?
- Arrow Theorem implies: If $F \neq$ dictator and f, g, h are non-constant then: $PDX(f) \geq 6^{-n}$.
- Notation: Write $D(F,G) = P(F(\sigma) \neq G(\sigma))$.
- Q: Suppose F is low influence or transitive and fair – what is the lowest possible probability of paradox?

Paradoxes and Stability

- Lemma 1 (Kalai 02):
- $PDX(F) = \frac{1}{4}(1 + E[f(x)g(y)] + E[f(x)h(z)] + E[g(y)h(z)])$
- Pf: Look at $s : \{-1,1\}^3 \rightarrow \{0,1\}$ which is 1 on $(1,1,1)$ and $(-1,-1,-1)$ and 0 elsewhere. Then
- $s(a,b,c) = \frac{1}{4}(1+ab+ac+bc)$.

- Note that (X,Y) is distributed as:
- $E[X_i] = E[Y_i] = 0$ and $E[X_i Y_i] = -1/3$.
- If F is fair then f,g,h are fair and we can write:
- $PDX(F) = \frac{1}{4}(1 - E[f(x)g(y)] - E[f(x)h(z)] - E[g(y)h(z)])$
- Where now (X,Y) is distributed as:
- $E[X_i] = E[Y_i] = 0$ and $E[X_i Y_i] = +1/3$



Paradoxes and Stability

- $PDX(F) = \frac{1}{4}(1 - E[f(x)g(y)] - E[f(x)h(z)] - E[g(y)h(z)])$
- Where now (X,Y) is distributed as:
- $E[X_i] = E[Y_i] = 0$ and $E[X_i Y_i] = +1/3$
- Fairness implies $E[f] = E[g] = E[h] = 0$.
- By majority is stablest
  $E[f(x)g(y)] < E[m_n(x) m_n(y)] + \epsilon$.

- Thm(Kalai 02): If F is fair and of max influence at most δ or transitive then:
- $PDX(F) > \lim PDX(Maj_n) - \varepsilon$ where $\varepsilon \to 0$ as $(\delta \to 0 / n \to \infty)$

Probability of a Paradox

- We already know that we cannot avoid paradoxes for low influence functions.
- Q: Can we avoid paradoxes with good probability for any non-dictatorial function?

Probability of a Paradox

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- Q: Can we avoid paradoxes with good probability with any non-dictatorial function?

- Let $f=g=h$ where $f(x) = x_1$ unless $x_2 = \ldots = x_n$ in which case $f(x) = x_2$.
- Non-dictatorial system.
- Paradox probability is exponentially small.

- Q (more reasonable): Is it true that the only functions with small paradox probability are close to dictator?

Probability of a Paradox

- Kalai-02: If IIA holds with $F = (f,g,h)$ and
- $E[f] = E[g] = E[h] = 0$ then
- $PDX(F) < \varepsilon \Rightarrow \exists$ a dictator i s.t.:
- $D(F,\sigma_i) < K\,\varepsilon$ or $D(F,-\sigma_i) < K\,\varepsilon$
- Where K is some absolute constant.
- Keller-08: Same result for symmetric distributions.

Probability of a Paradox

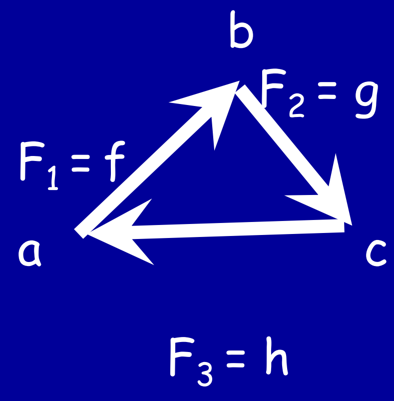
- Thm M-10: $\forall \varepsilon, \exists \delta$ s.t.:
- If IIA holds with $F = (f,g,h)$ and
- $\max \{|E[f]|, |E[g]|, |E[h]|\} < 1-\varepsilon$ and
- $\min_i \min \{D(F,\sigma_i), D(F,-\sigma_i)\} > \varepsilon$
- Then $P(F) > \delta$.

- General Thm M-10: $\forall k, \varepsilon \exists \delta$ s.t.:
- If IIA holds for F on k alternatives and
- $\min \{D(F,G) : G \in F_k(n)\} > \varepsilon$
- Then: $P(F) > \delta$.

- Comment: Can take $\delta = k^{-2} \exp(-C/\varepsilon^{21})$

A Quantitative Lemma

- Def: The influence of voter 1 on f (denoted $I_1(f)$) is:
- $I_1(f) := P[f(-,x_2,\dots,x_n) \neq f(+,x_2,\dots,x_n)]$
- Lemma (M-09): Any constitution $F=(f,g,h)$ on 3 alternatives which satisfies IIA and has
- $I_1(f) > \varepsilon$ and $I_2(g) > \varepsilon$
- Satisfies $PDX(F) > \varepsilon^3/36$.



- Pf:
- Let $A_f = \{x_3,\dots,x_n : 1 \text{ is pivotal for } f(*,*,x_3,\dots,x_n)\}$
- Let $B_g = \{y_3,\dots,y_n : 2 \text{ is pivotal for } g(*,*,y_3,\dots,y_n)\}$
- Then $P[A_f] > \varepsilon$ and $P[B_g] > \varepsilon$
- By "Inverse Hyper-Contraction": $P[A_f \cap B_g] > \varepsilon^3$.
- By Lemma: $PDX[F] \geq 1/36\, P[A_f \cap B_g] > \varepsilon^3/36$.

Inverse Hyper Contraction

- Note: (x_i,y_i) are i.i.d. with $E(x_i,y_i) = (0,0)$ and $E[x_i\, y_i] = -1/3$
- Results of C. Borell 82: $\Rightarrow$
- Let $f,g : \{-1,1\}^n \to R_+$ then
- $E[f(x)\, g(y)] \geq |f|_p\, |g|_q$ if $1/9 \leq (1-q)\,(1-p)$ and $p,q < 1$.
- In particular: taking f and g indicators obtain:
- $E[f] > \varepsilon$ and $E[g] > \varepsilon \Rightarrow E[fg] > \varepsilon^3$.
- Implications in: M-O'Donnell-Regev-Steif-Sudakov-06.

- Note: "usual" hyper-contraction gives:
- $E[f(x)\, g(y)] \leq |f|_p\, |g|_q$ for all functions if
- $(p-1)(q-1) \geq 1/9$ and $p,q>1$.

Inverse Hyper Contraction

The Use of Swedish Technology

IKEA Store Falls Apart! Experts Blame Cheap Parts, Confusing Blueprint
From SD Headliner, Mar 25, 09.

Quantitative Arrow – 1st attempt

- Thm M-10: $\forall\ \varepsilon, \exists\ \delta$ s.t if IIA holds with $F = (f,g,h)$ &
- $\max\{|E[f]|, |E[g]|, |E[h]|\} < 1-\varepsilon$ &
- $\min\{D(F,G) : G \in F_3(n)\} > 3\varepsilon$
- Then $PDX(F) > (\varepsilon/96n)^3$.

- Pf Sketch: Let $P_f = \{i : I_i(f) > \varepsilon\, n^{-1}/4\}$
- Since $\sum I_i(f) > Var[f] > \varepsilon/2$, P_f is not empty.
- If there exists $i \neq j$ with $i \in P_f$ and $j \in P_g$ then $PDX(F) > (\varepsilon/96n)^3$ by quantitative lemma.
- So assume $P_f = P_g = P_h = \{1\}$ and $P(F) < (\varepsilon/96n)^3$
- $\Rightarrow D(f, \pm x_i) \leq \varepsilon$ or $D(f, \pm 1) \leq \varepsilon$ (same for g and h)
- $\Rightarrow D(F,G) \leq 3\,\varepsilon$ where $G(\sigma) = G(\sigma_1)$.
- $PDX(G) \leq 3\,\varepsilon + (\varepsilon/96n)^3 < 1/6 \Rightarrow G \in F_3(n)$.

Quantitative Arrow – Real Proof

- Pf High Level Sketch:
- Let $P_f = \{i : I_i(f) > \varepsilon\}$.
- If there exists $i \neq j$ with $i \in P_f$ and $j \in P_g$ then $PDX(F) > \varepsilon^3 / 36$ by quantitative lemma.
- Two other cases to consider:
- I. $P_f \cap P_g = P_f \cap P_h = P_g \cap P_h$ is empty
- In this case: use Invariance + Gaussian Arrow Thm.
- II. $P_f \cup P_g \cup P_h = \{1\}$.
- In this case we condition on voter 1 so we are back in case I.

Quantitative Arrow – Real Proof

- The Low Influence Case:
- We want to prove the theorem under the condition that $P_f \cap P_g = P_f \cap P_h = P_g \cap P_h$ is empty.
- Let's first assume that $P_f = P_g = P_h$ is empty – all functions are of low influence.
- Recall:
- $PDX(F) = \frac{1}{4} (1 + E[f(x)g(y)] + E[f(x)h(z)] + E[g(y)h(z)])$
- Where now (X,Y) is distributed as:
- $E[X_i] = E[Y_i] = 0$ and $E[X_i Y_i] = -1/3$
- By a version of Maj-Stablest Majority is Stablest:
- $PFX(F) > PDX(u,v,w)$ + error(I) where
- $u(x) = sgn(\sum x_j + u_0)$ and $E[u] = E[f]$ etc.

Quantitative Arrow – Real Proof

- By Majority is Stablest:
- PFX(F) > PDX(u,v,w) + error(I) where
- $u(x) = sgn(\sum x_j + u_0)$ and $E[u] = E[f]$ etc.
- Remains to bound PDX(u,v,w)
- By CLT this is approximately:
- $P[U>0,V>0,W>0] + P[U<0, V<0, W<0]$ where $U \sim N(E(u),1)$, $V \sim N(E(v),1)$ and $W \sim N(E(w),1)$ &
- $Cov[U,V] = Cov[V,W] = Cov[W,U] = -1/3$.
- For Gaussians possible to bound.

Quantitative Arrow – Real Proof

- In fact the proof works under the weaker condition that $P_f \cap P_g = P_f \cap P_h = P_g \cap P_h$ is empty.
- The reason is that the strong version of majority is stablest (M-10) says:
- If $\min(I_i(f), I_i(g)) < \delta$ for all i and u and v are majority functions with E[f]=u, E[g] = v then:
- $E[f(X)\, g(Y)] < \lim n\, E[u_n(X)\, v_n(Y)] + \varepsilon(\delta)$ where
- $\varepsilon(\delta) \to 0$ as $\delta \to 0$.

Probability of a Paradox for Low Inf Functions

- Thm: (Follows from MOO-05): $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. If
- $\max_i \max\{I_i(f), I_i(g), I_i(h)\} < \delta$
  then $PDX(F) > \lim_{n \to \infty} PDX(f_n, g_n, h_n) - \epsilon$
- where $f_n = sgn(\sum_{i=1}^n x_i - a_n)$, $g_n = sgn(\sum_{i=1}^n y_i - b_n)$, $h_n = sgn(\sum_{i=1}^n z_i - c_n)$ and a_n, b_n and c_n are chosen so that $E[f_n] \sim E[f]$ etc.
- Thm (Follows from M-08): The same theorem holds with $\max_i 2^{nd}(I_i(f), I_i(g), I_i(h)) < \delta$.
- So case I. of quantitative Arrow follows if we can prove Arrow theorem for threshold functions.
- (Recall case I.: $P_f \cap P_g = P_f \cap P_h = P_g \cap P_h$ is empty)
- Pf for "threshold functions" using Gaussian analysis.

Pf of Majority is Stablest

- Majority is Stablest Conj: If $E[f] = E[g] = 0$ and f,g have all influences less than δ then
  $E[f(x)g(y)] > E[m_n(x)\, m_n(y)] - \epsilon$.
- Ingredients:
- I. Thm (Borell 85): (N_i, M_i) are i.i.d. Gaussians with
- $E[N_i] = E[M_i] = 0$ and $E[N_i\, M_i] = -1/3$, $E[N_i^2] = E[M_i^2] = 1$ and f and g are two functions from R^n to $\{-1,1\}$ with $E[f] = E[g] = 0$ then:
- $E[f(X)\, g(Y)] \geq E[sgn(X_1)\, sgn(Y_1)]$.
- By the CLT: $E[sgn(X_1)\, sgn(Y_1)] = \lim_{n \to \infty} E[m_n(x)\, m_n(y)]$
- II. Invariance Principle [M+O'Donnell+Oleszkiewicz(05)]:
- Gaussian case $\Rightarrow$ Discrete case.

The Geometry Behind Borell's Result

- I. Thm (Borell 85): (N_i, M_i) are i.i.d. Gaussians with
- $E[N_i] = E[M_i] = 0$ and $E[N_i M_i] = -1/3$, $E[N_i^2] = E[M_i^2] = 1$ and f and g are two functions from R^n to $\{-1,1\}$ with $E[f] = E[g] = 0$ then:
- $E[f(X)\, g(Y)] \geq E[sgn(X_1)\, sgn(Y_1)]$.

- Spherical Version: Consider $X \in S^n$ uniform and $Y \in S^n$ chosen uniformly conditioned on $\langle X,Y \rangle \leq -1/3$.
- Among functions f,g with $E[f] = E[g] = 0$ what is the minimum of $E[f(X)\, g(Y)]$?
- Answer: $f = g$ = same half-space.

The Geometry Behind Borell's Result

- More general Thm (Isaksson-M 09): $(N^1,\ldots,N^k)$ are k n-dim Gaussain vectors $N^i \sim N(0,I)$.
- $Cov(N^i,N^j) = \rho I$ for $i \neq j$, where $\rho > 0$.
- Then if $f_1,\ldots,f_k$ are functions from R^n to $\{0,1\}$ with $E[f] = 0$ then:
- $E[f_1(N^1) \ldots f_k(N^k)] \leq E[sgn(N^1_1) \ldots sgn(N^k_1)]$

- Proof is based on re-arrangements inequalities on the sphere.
- Gives that majority maximizes probability of unique winner in Condercet voting for low influence functions.

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HW 1

- Let $f=g=h$ be the $m \times m$ electoral college and consider IIA vote with $F=(f,g,h)$.
- Given a uniform vote x and y obtained from x by a single uniformly chosen voting error, what is
  $\lim P[f(x) \neq f(y)] \times m$ as $m \to \infty$.
- Assume x is obtained from y by flipping each coordinate with probability ε independently.
  What is $\lim P[f(x) \neq f(y)]$ as $m \to \infty$
- What is the limiting probability of an Arrow paradox assuming uniform voting and $m \to \infty$?

HW 2

- Consider the function $\Psi(f,i)$ which given
- a function $f : \{-1,1\}^n \rightarrow \{-1,1\}$ and a voter i returns an x s.t. f is pivotal on x and voter i. The function returns Null if no such x exist.
- Given access to $\Psi(f, ?)$ $\Psi(g, ?)$ and $\Psi(h, ?)$ Design an efficient algorithm that decides if (f,g,h) has a non-transitive outcome and if such an outcome exist it produces it. The running time of the algorithm should be linear in n.

HW 2 – continued

- Assume that the functions f,g and h are monotone and submodular so that for all x,y:
- $f(\min(x,y)) + f(\max(x,y)) \leq f(x) + f(y)$

where the maximum is taken coordinate-wise.

Show that the problems of deciding if all outcome of (f,g,h) are transitive and finding a non-transitive outcomes if such exist can both be solve in linear time (assuming access to f,g and h takes one unit of time)

HW 3

- Consider the 3-reursive majority functions f_n:
- $f_1(x(1),x(2),x(3)) = maj(x(1),x(2),x(3))$
- $f_{k+1}(x(1)\ldots,x(3^{k+1})) = maj(f_1(x),f_2(y),f_3(z))$ where
- $x= (x(1),\ldots,x(3^k)), y = (x(3^{k+1}+1),\ldots,x(2*3^k)), z= \ldots$
- Let (x,y) be uniform with y different from x in one coordinate. What is $P[f_k(x) = f_k(y)]$?
- Assuming x is uniform and y is obtained from x by flipping each coordinate with probability ε, show:
- $P[f_k(x) = f_k(y)] = \frac{1}{2} + 3^{k\alpha + o(1)}$ for some α. Find $\alpha(\varepsilon)$
- Consider ranking using F=(f(x),f(y),f(z)). What is the limit of $P[F(\sigma)$ is non-transitive$]$? What is the next order term (both as $k \to \infty$)

HW 4

- Consider the Plurality coordination problem on a social network where initially each player receives one of 3 colors.
- Design a protocol using the color and one extra bit of memory that reachs coordination.

HW 5

- Consider the voter model on $G=(V,E)$.
- Assume that the model is run for k different topics and that further
- Assume that for each topic k, time t and all $v \in V$ the opinion of v at topic k denoted $v(k,t)$ is known but:
- The graph E is not known.
- Design an algorithm that finds the edges of the graph G from the record of the votes.
- How large should k and t be for the algorithm to have a high probability of recovering G?