Errors in Binary Voting

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Draft

Unbiased Signals

- In the next few weeks we will deal with the case of unbiased signals / uniformed voters.
- Why do we do it?

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- Why do we do it?
- These measures provide “stress-test” for the voting methods we are using:
- If voters all have strong correlated opinions then:
- Small effects of (small) errors in the voting scheme
- Will not see irrational outcomes.
- Outcome hard to manipulate.

Unbiased Signals

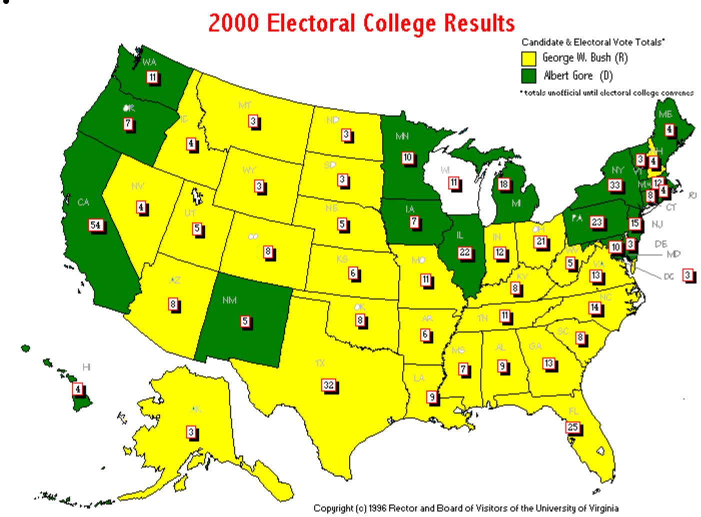
- What are unbiased signals / uniformed voters?

Unbiased Signals

- What are unbiased signals / uniformed voters?
- Worst case scenarios: exists voting configurations resulting in errors/manipulation/etc.
- Average case scenarios: On average there is a good probability of errors/manipulation/etc.
- Average with respect to the most uniformed measure = the uniform measure.

Definition of voting schemes

- Today topic is errors of voting schemes on binary decisions.
- A population of size n is to choose between two options / candidates.
- A voting scheme is a function that associates to each configuration of votes which option to choose.
- Formally, a voting scheme is a function $f : \{-1,1\}^n \to \{-1,1\}$.
- Two prime examples:
  - Majority vote,
  - Electoral college.



Properties of voting schemes

- Some properties of voting schemes:
- We will always assume that candidates are treated equally:

  The function f is fair: if $f(-x) = -f(x)$.
- We always assume that stronger support in a candidate shouldn't heart her:

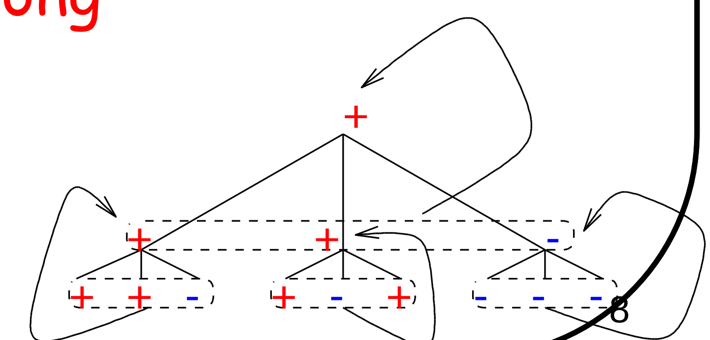
  The function f is monotone: $x \geq y \Rightarrow f(x) \geq f(y)$, where $x \geq y$ if $x_i \geq y_i$ for all i.
- Note that both majority and the electoral college are anti-symmetric and monotone.

Democracy and voting schemes

- Two interpretations of democracy:
- "Weak democracy" – each voter has the same power: There exists a transitive group $\Gamma \subset S_n$ such that for all $\sigma \in \Gamma$ and all x it holds that

$$f((x_{\sigma(i)})) = f((x_i)) \quad (***)$$

- "Strong democracy" – each set of voters has the same power – $(***)$ holds for all $\sigma \in S_n$.
- Easy: Monotonicity + fairness + strong democracy $\Rightarrow$ f = majority.
- But: Electoral college is weak democracy (mathematically)



Errors in voting

- **Claim:** Any non-constant voting scheme is prone to errors.
- Pf: Since it is not constant there exist x and y such that $f(x) \neq f(y)$.

- **Claim:** Any non-constant voting scheme is prone to an error of a single voter.

- Pf: Otherwise whenever we change a single coordinate the value of f stays the same. But this means that any # of coordinates changes does not change the value of f. .

To the uniform measure

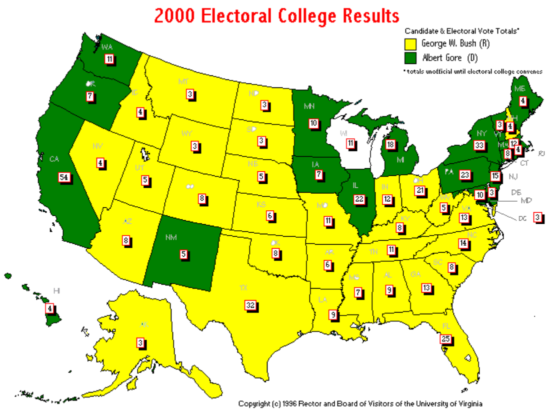
- Assume x is chosen uniformly in $\{-1,1\}^n$.
- Let $y = N_\varepsilon(x)$ is obtained from x by flipping each of x coordinates with probability ε.
- Question: What is the probability that the population voted for who they meant to vote for?
- What is $S_f(\varepsilon) = P[f(x) = f(y)]$?
- Which is the most sensitive / stable f?
- Is there an f which is both stable and sensible?
- Maj? Elctoral college?

Stability without democracy

- $N(x, y) = P[N_\varepsilon(x) = y]$, $\eta = 1 - 2\varepsilon$ and $Z(f,\eta) := \langle f,Nf\rangle = E[f(x)\, f(y)]$
- $S(f,\varepsilon) = (Z(f,\eta)+1)/2$
- N has the eigenvectors $u_S(x) = \prod_{i \in S} x_i$, corresponding to the eigenvalues $\eta^{|S|}$.
- $P[f(x) = f(N_\varepsilon(x))] = \frac{1}{2} + E[f(x)f(N_\varepsilon(x))]/2 =$
  $= \frac{1}{2} + \langle f,Nf\rangle/2$.
- Write $f(x) = \sum_S f_S\, u_S(x)$.
- Since $\langle f,1\rangle = 0$, $\langle f,Nf\rangle = \sum_{S \neq \emptyset} f_S^2\, \eta^{|S|} \leq \eta$ and therefore $S_f(\varepsilon) = 1 - \varepsilon$.
- Dictatorship, $f(x) = x_i$ is the only optimal function.

Stability with democracy

- How stable can we get with democracy?
- Thm Sheffield (1899):
- $\lim_{n \to \infty} Z(f,\eta) = \arcsin(\eta)/\pi$ for $f = maj_n$.
- When ε is small $S_f(\varepsilon) \sim 1 - 2\,\varepsilon^{1/2}/\pi$.
- Pf?



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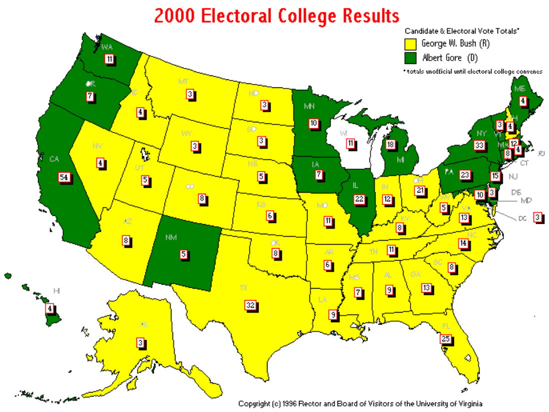

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- Pf: Let $N = n^{-1/2} \sum x_i$, $M = n^{-1/2} \sum y_i$
- By CLT $E[f(x)\ f(y)] \to E[sgn(N)\ sgn(M)] =$

  $= 1 - 2\ P[sgn(N) \neq sgn(M)] =$

  $= 1 - P[N > 0, M < 0]$.
- Write $M = a\ N - b\ U$ where $a = \eta$ and $a^2 + b^2 = 1$ and N are independent. Then
- $P[N > 0, M < 0] = P[0 < N < (b/a)U] = \arctan(b/a)/2\ \pi$
- Why?
- This give the result using trigonometric identities.

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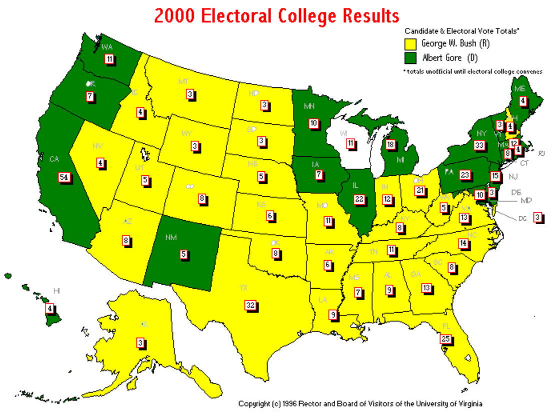
- When ε is small $S_f(\varepsilon) \sim 2\, \varepsilon^{1/2}/\pi$.
- Claim: An $n^{1/2} \times n^{1/2}$ electoral college gives $S_f(\varepsilon) = \Theta(\varepsilon^{1/4})$.
- Why?



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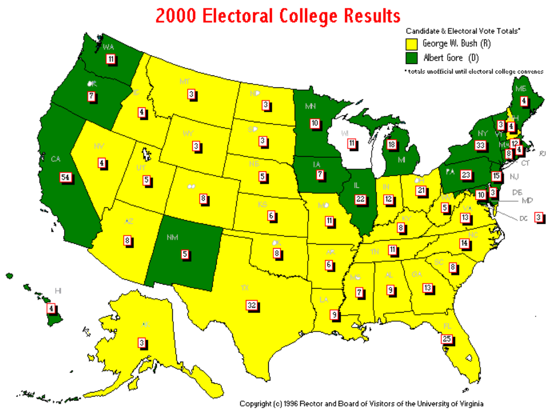

- Thm (Majority is Stablest; M-O'Donnell-Olseskiwsz): If $f = f_n$ satisfies
- $\max \{e_i(f) : 1 \le i \le n\} = o(1)$, then
- $\lim_{n \to \infty} S_f(\varepsilon) \ge \frac{1}{2} - \arcsin(1 - 2\varepsilon)/\pi$.
- $\Rightarrow$ most stable "weak democracy" = maj.
- Won't do proof. Can hear a bit about it tomorrow at 4pm.

Majority is Least Stable

- Interestingly if we are just interested in a single error then:
- Thm:
- Among all monotone functions, majority maximizes the probability $P[f(x) \neq f(y)]$ where y is obtained from x by a random flip of one bit.
- Pf: We want to maximize:
- $\sum \{f(y)-f(x) : y \text{ directly above } x\} =$
- $\sum_{k=0}^{n} \sum \{(k - (n-k))f(x) : x \text{ s.t. } \#(1,x) = k\} =$
- $\sum_{k=0}^{n} \sum \{(n-2k)f(x) : x \text{ s.t. } \#(1,x) = k\}$.

- So Majority is least stable for fixed flip probability and most stable for flip probability << 1/n.

Getting sensitive

- How sensitive can a fair monotone functions be?
- Interesting in learning, neural networks, hardness amplification …
- maj_n maximizes the isoperimetric edge bounds among all monotone functions and $I(maj_n)^2 \sim 2n/\pi$.
- By Russo's formula: $I(f) = Z'(f,1)$.
- But $Z'(f,1) = \sum_S |S| f_S^2$.
- Consider the following relaxation of the problem: minimize $\sum_S a_S \eta^{|S|}$ under the constraints:
  - $\sum_S a_S = 1$, $a_S \geq 0$, $\sum_S |S| a_S \sim\leq (2n/\pi)^{1/2} := \alpha$
  - We get that $Z(f,\eta) \sim\geq \eta^{\alpha}$
  - In particular, any monotone function requires randomly flipping at least order $n^{1/2}$ votes to have probability > 0.00001 to flip the elections.

Getting sensitive

- Kalai: Are there any functions that are so sensitive?
- Kalai: Is it enough to flip $n^{1/2}$ of the votes in order to flip outcome with probability $\Omega(1)$?
- Thm[M-O'Donnell]:
- rec-maj-k satisfies $\langle f, Nf \rangle \sim\leq \eta^{\alpha(n,k)}$ where $\alpha(n,k) \sim n^{\beta(k)}$ and $\beta(k) \to \frac{1}{2}$ as $k \to \infty$ (enough to flip $n^{1-\beta(k)}$)
- rec-maj with increasing arities gives that it is enough to flip $\log^t(n) * n^{1/2}$ where $t = \frac{1}{2} \log_2(\pi/2)$.
- Talgrand's random function gives that it is enough to flip $c\, n^{1/2}$.