Manipulation & GS Theorem

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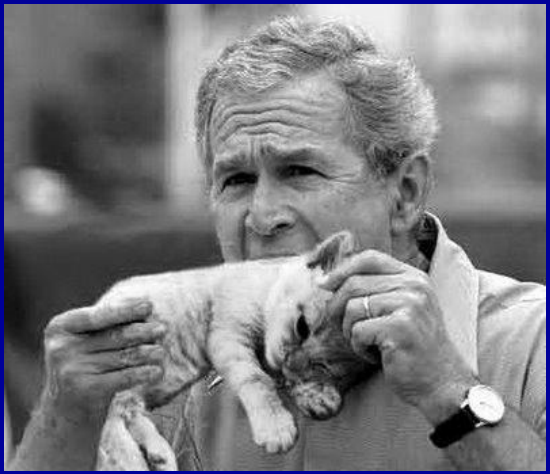
Truthfulness in Binary Voting

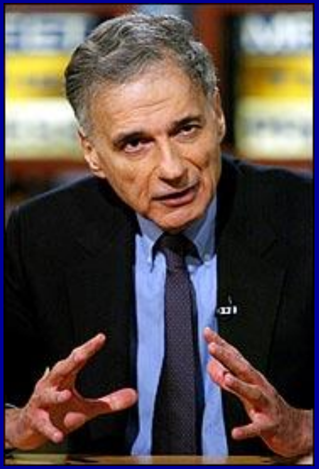
- n voters to vote if $+$ or $-$.
- $x_i \in \{+,-\}$ is voter i'th vote.
- Outcome = $f(x_1,\dots,x_n)$, where
- $f : \{-,+\}^n \rightarrow \{-,+\}$
- Def: f is manipulable by voter 1 if there exists $x_2,\dots,x_n$ such that:
- $f(+,x_2,\dots,x_n) = -$, $f(-,x_2,\dots,x_n) = +$.
- Which f cannot be manipulated by any voter?

Manipulation and Montonicity

- Def: f is manipulable by voter 1 if there exists $x_2,\ldots,x_n$ such that: $f(+,x_2,\ldots,x_n) = -$, $f(-,x_2,\ldots x_n) = +$.
- Which f are non-manipulable?
- Claim: f is manipulable if and only if f is not monotone.
- Recall: f is monotone if $\forall$ i, $x_i \geq y_i \Rightarrow f(x_1,\ldots,x_n) \geq f(y_1,\ldots,y_n)$.

Manipulation: 3 or more alt.



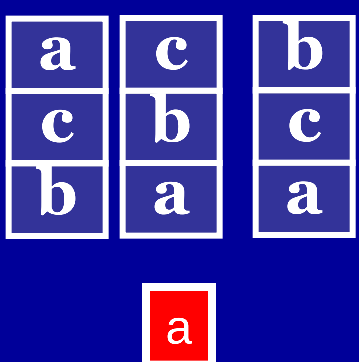


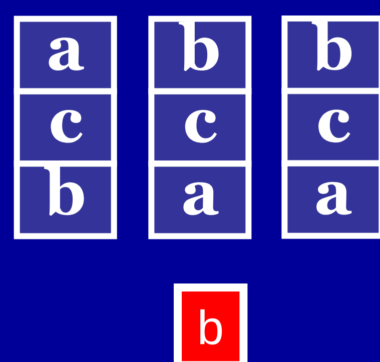
a	b	c
c	c	b
b	a	a
45	40	15

- Last group of voters could manipulate in Plurality vote.

Manipulation by a Single Voter

- n people rank 3 alternatives.
- Plurality winner = most frequently ranked at top.
- (if tied go according to first voter).
- <u>Example:</u> If second voter knows the preferences of all voters will prefer to vote differently than her true preference.
- <u>Question:</u> Is this avoidable?



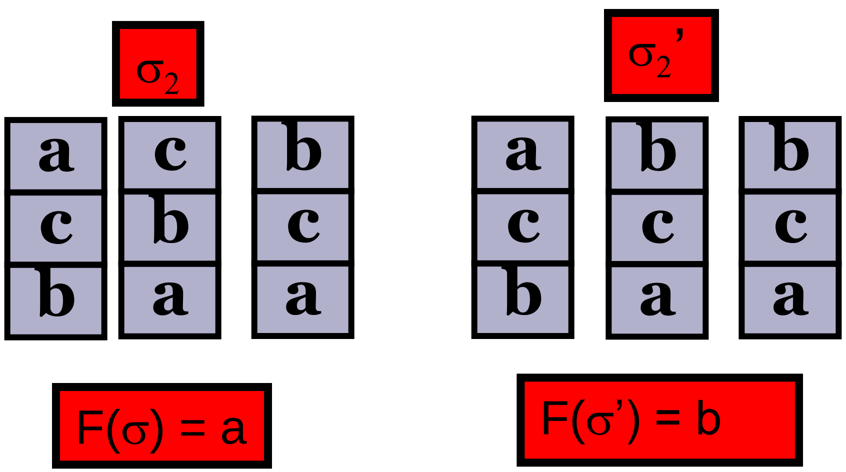


Choice Functions and Manipulation

Definition: F is a social choice function if F associates to each collection of n rankings a winner:

$F : S(A,B,\ldots,K)^n \rightarrow \{A,B,C,D,\ldots,K\}$'

Definition: F is manipulable by voter i if there exists two rankings $\sigma = (\sigma_{i,}\ , \sigma_{-i})$, $\sigma' = (\sigma'_{i,}\ , \sigma_{-i})$, s.t. $\sigma_i(F(\sigma')) > \sigma_i(F(\sigma))$ (Voter i with preference σ_i would prefer outcome $F(\sigma')$)



Example: Manipulation by voter 2

Examples of non-manipulable Fs

- The "dictator" $F(\sigma) = top(\sigma_i)$ is non-manipulable.
- A function $F : S(A,B)^n \rightarrow \{A,B\}$ is non-manipulable if and only if F is monotone.
- Are there other examples?

- Def: F is Neutral if for all σ' in $S(A,B,\ldots,K)$ and σ in $S(A,B,\ldots,K)^n$ it holds that: $F(\sigma' \sigma) = \sigma' F(\sigma)$
- In words: Fair among all alternatives.
- Def: F satisfies Unanimity if

  $top(\sigma_1) = \ldots = top(\sigma_n) = a \Rightarrow F(\sigma) = a$
- Def: Non-manipulable = strategy-proof.

Gibbard–Satterthwaite Thm

- Thm (Gibbard-Satterthwaite 73,75):
  If F ranks at least 3 alternatives,
- satisfies unanimity / is onto &
- is strategy proof

Then F is a dictator

. We'll follow proofs in to Lars Gunar Svensson – 99

Two Simple Lemmas

- Lemma 1 (Monotonicity):
- If F is strategy proof and $F(\sigma) = a$ and τ satisfies that for all x and all i:
- $\sigma_i(a) \geq \sigma_i(x) \Rightarrow \tau_i(a) \geq \tau_i(x)$
- then $F(\tau) = a$.

Two Simple Lemmas

- Lemma 1 (Monotonicity):
- If F is strategy proof and $F(\sigma) = a$ and τ satisfies that for all x and all i:
- $\sigma_i(a) \geq \sigma_i(x) \Rightarrow \tau_i(a) \geq \tau_i(x)$
- then $F(\tau) = a$.
- Pf: Suffices to prove when $\tau_i = \sigma_i$ for $i>1$.
- Assume by contradiction that $a \neq b = F(\tau)$ then from strategy-proofness $\sigma_1(b) \leq \sigma_1(a)$
- therefore $\tau_1(b) \leq \tau_1(a)$ but then voter 1 will prefer to use σ_1.
-

Two Simple Lemmas

- <u>Lemma 2 (Pareto):</u>
- Assume that F is onto and strategy-proof.
- Let σ satisfy that $\sigma_i(a) > \sigma_i(b)$ for all i.
- Then $F(\sigma) \neq b$.

Two Simple Lemmas

- Lemma 2 (Pareto):
- Assume that F is onto and strategy-proof.
- Let σ satisfy that $\sigma_i(a) > \sigma_i(b)$ for all i.
- Then $F(\sigma) \neq b$.
- Pf: Assume $F(\sigma) = b$.
- Since F is onto there exists a τ with $F(\tau) = a$.
- Let σ'_i put b then a then like in σ.
- Monotonicity lemma implies that $F(\sigma') = F(\sigma) = b$.
- Monotonicity lemma also implies that $F(\sigma') = F(\tau) = a$.

Proof in the case of two voters

- Pf:
- Let $u := a>b>others$ and $v := b>a>others$.
- We know that $f(u,v)$ is either a or b. Let's assume it's a.
- $\Rightarrow$ for every v' which has b at the top we have $f(u,v') = a$ in particular for v' which has a at the bottom.
- $\Rightarrow$ (by monotonicity lemma) $f(u',v') = a$ for all u' which has a on top.
- Let A_1 be alt. a such that if they are at the top of u outcome is a and similarly A_2. Then clearly $A_1 \cap A_2 = \text{empty}$
- $\Rightarrow f(u,v) = top(u)$ as needed.

Reduction to two voters

- Lemma:
- It suffices to prove the GS theorem for the case of two voters.
- Pf: By induction on the number of voters n. For general n define $g(u,v) = f(u,v,v,v,v,,v,v)$. Note that:
- Lemma 2 $\Rightarrow$ g is Pareto.
- We next argue that if f is strategy proof so is g. Otherwise there are u,v,v' s.t. $v(g(u,v')) > v(g(u,v))$.
- Define $u_k = (u, k \times v', (n-k-1) \times v)$
- We must have a k where $v(g(u_{k+1}) > v(g(u_k))$
- $\Rightarrow$ g is a strategy proof $\Rightarrow$ g is a dictator.

Reduction to two voters – cont.

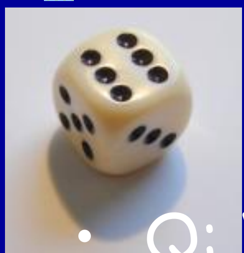
- Pf: $g(u,v) = f(u,v,v,v,v,,v,v)$ is a dictator.
- If it is dictator on voter 1 – then monotonicty Lemma 1 f is also a dictator on voter 1.
- So assume g is a dictator on voter 2.
- Fix u^* and look at $h(v_2,\dots,v_n) = f(u^*,v_2,\dots,v_n)$
- The h is onto and strategy proof so it is dictatorial.
- WLOG assume 2 is the dictator and fix $v_3,\dots,v_n$.
- Then $z(u,v) = f(u,v,v_3,\dots,v_n)$ is onto and strategy proof and 1 cannot be the dictator.
- So z is a dictator on voter 2 $\Rightarrow$ f is dictator on voter 2.

Gibbard–Satterthwaite Thm

- Thm (Gibbard-Satterthwaite 73,75):
  If F ranks at least 3 alternatives,
- satisfies unanimity (or is onto) &
- is non-manipulable then
  Then F is a dictator.

- Let $D_k(n)$ = {dictators on k alt and n voters}
- GS Thm: If F is Neutral & Non Manipulable $\Rightarrow F \in D_k(n)$
- More generally:
  F depends on two voters & Takes at least 3 values $\Rightarrow$ F is manipulable.

Random Rankings:



- Kelly 95 : Consider people voting according to a random order on $\{A,\ldots,K\}$ = uniformly in S_K^n

- Q: What is the probability of a manipulation:
- Def: $M(F) = P[\sigma$: some voter can manip F at $\sigma]$.

- GS Thm: If not in $D_k(n)$ then:
- $M(F) \geq (k!)^{-n}$.

  If manipulation so unlikely perhaps do not care?

- Notation: Write $D(F,G) = P(F(\sigma) \neq G(\sigma))$.

  $D(F,D_k(n)) = \min \{ D(F,G) : G \in D_k(n) \}$

High Probability Manipulation

- Q:

- Is it true that for all eps exists a delta s.t.
- if F is neutral and
- $D(F,D_k(n)) > \varepsilon$ then $P(F \text{ manipulable}) > \delta$?

High Probability Manipulation

- Q:
- Is it true that if F is neutral and
- $D(F,D_k(n)) > \varepsilon$ then $P(F \text{ manipulable}) > \delta$?

- A: No
- Example: Plurality function

High Probability Manipulation

- Thm Issakson-Kindler-M-10:
- If F is Neutral and $k \geq 3$ then $M(F) \geq n^{-3} k^{-10} D(F, D_k(n))^2$
- Moreover: the trivial random algorithm manipulates with probability at least $n^{-3} k^{-10} D(F, D_k(n))^2$.

Related Work

- Bartholdi, Orlin (91), Bartholdi,Tovey Trick (93):
  Manipulation for a voter for some voting schemes is NP hard (for large k).

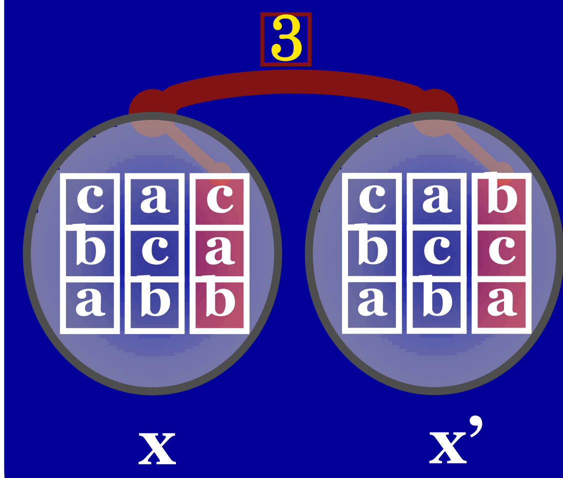
- Conitzer, Sandholm (93, 95) etc. : Hard on average?

- Conj (Friedgut-Kalai-Nisan 08): Random manipulation gives $M(F) \geq \text{poly}(n,k)^{-1}$. In particular easy on average.

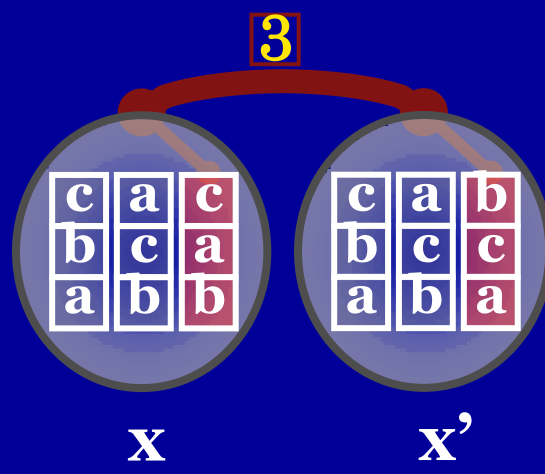
- Thm (FKN 08): For $k=3$ alternatives, and neutral F, it holds that $M(F) \geq n^{-1} D(F_k(n),D)^2$
  (no computational consequences)

Idea 1: The rankings graph

- We consider the graph with vertex set $S(A,B,\ldots K)^n$
- $e=[x,x']$ is an edge on voter i, if $x(j) = x'(j)$ for $j \neq i$ and $x(i) \neq x'(i)$.
- For $F : S(A,\ldots K)^n \to \{A,\ldots,K\}$, we call $e=[x,x']$ a boundary edge if $F(x) \neq F(x')$.



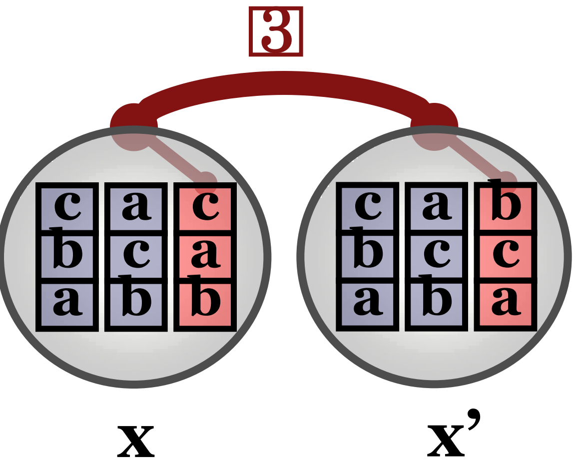
[x,x’] is an edge on voter 3



If F(x) = c and F(x’) = a then [x,x’] is a boundary edge

Write:
$e \in \partial_3[c,a]$

3 Types of Boundary edges



F(x) = a F(x’) = b

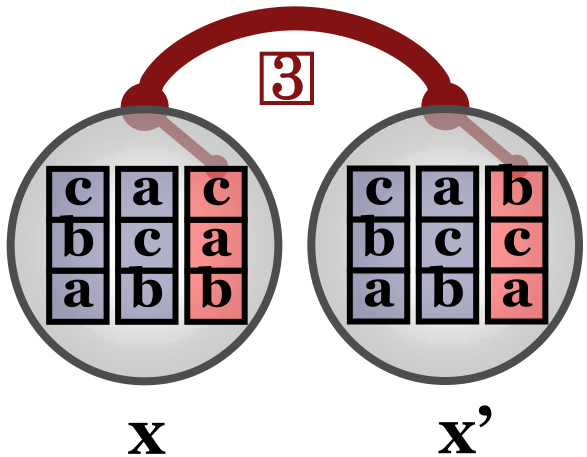
This edge is

monotone

and non-manipulable:

x ranks a above b

x’ ranks b above a



F(x) = a F(x’) = c

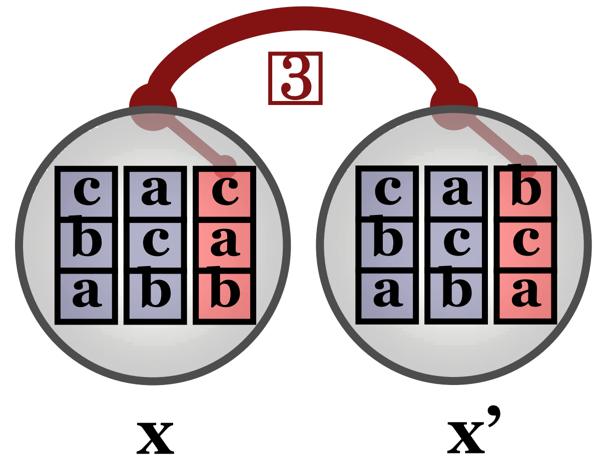
This edge is

monotone-neutral

and manipulable:

same order of

a,c in x,x’



F(x) = b F(x’) = c

This edge is

anti-monotone

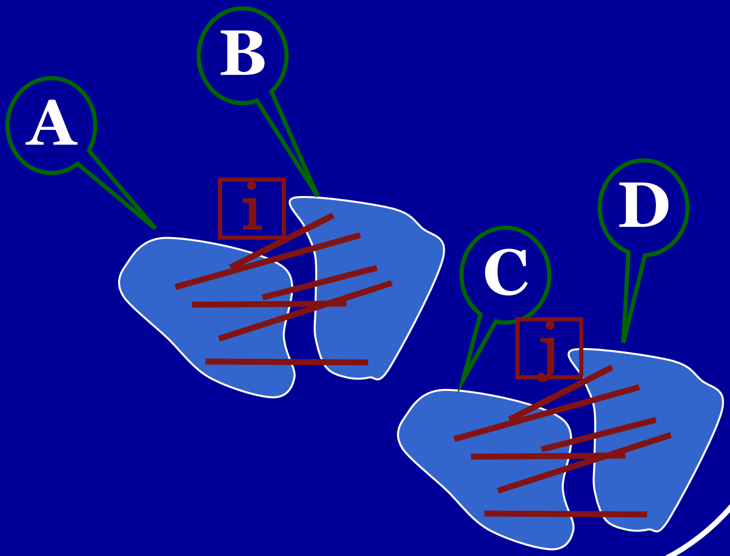
and manipulable:

x ranks c above b

x’ ranks b above c

Idea 2: Isoperimetry

- Assume 4 alternatives, unif. distribution.
- An Isoperimetric Lemma:
- If F is ε far from all dictators and Neutral
- Then there exists voters $i \neq j$ and s.t:
- $P[e \in \partial_i[A,B]] \geq \varepsilon\,(6n)^{-2}$, $P[e \in \partial_j[C,D]] \geq \varepsilon\,(6n)^{-2}$

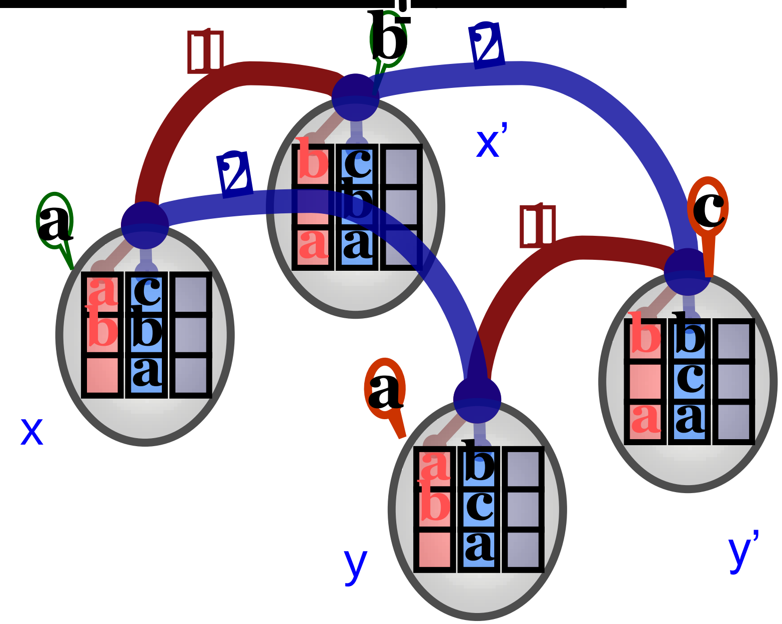


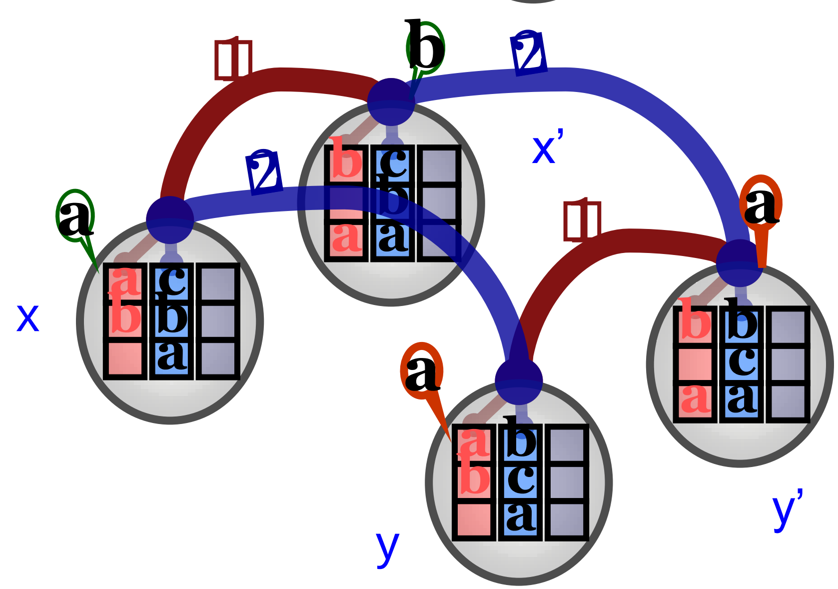
Idea 3: Paths and Flows on $\partial_i(A,B)$

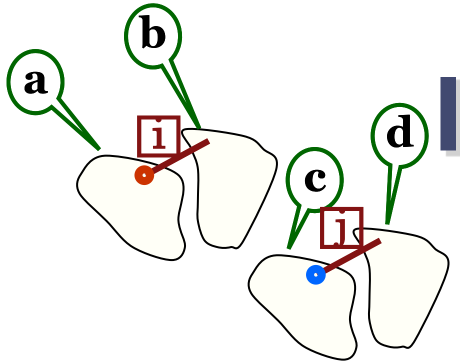
- Key Property: The space $\partial_i[A,B]$ is "nice":
- One can define "flows" and "paths" on it.
- &: $\partial\ \partial_i[A,B]$ "=" Manipulation points.
- Lemma: Let $[x,x'] \in \partial_i[A,B]$, $j \in [n]\backslash\{i\}$

  $y_{-j} = x_{-j}$ and $y'_{-j} = x'_{-j}$

  y_j, y'_j have same A,B order as x_j, x'_j
- Then either $[y,y'] \in \partial_i[A,B]$ or
- $\exists$ a manipulation point identical to x except in at most 3 voters.
- Pf: If $F(y)$ not in $\{A,B\}$
- apply GS fixing all voters but i,j.
- If $F(x) = F(y) = F(y') = A$, $F(x') = B$ then (x',y') is manipulation edge.





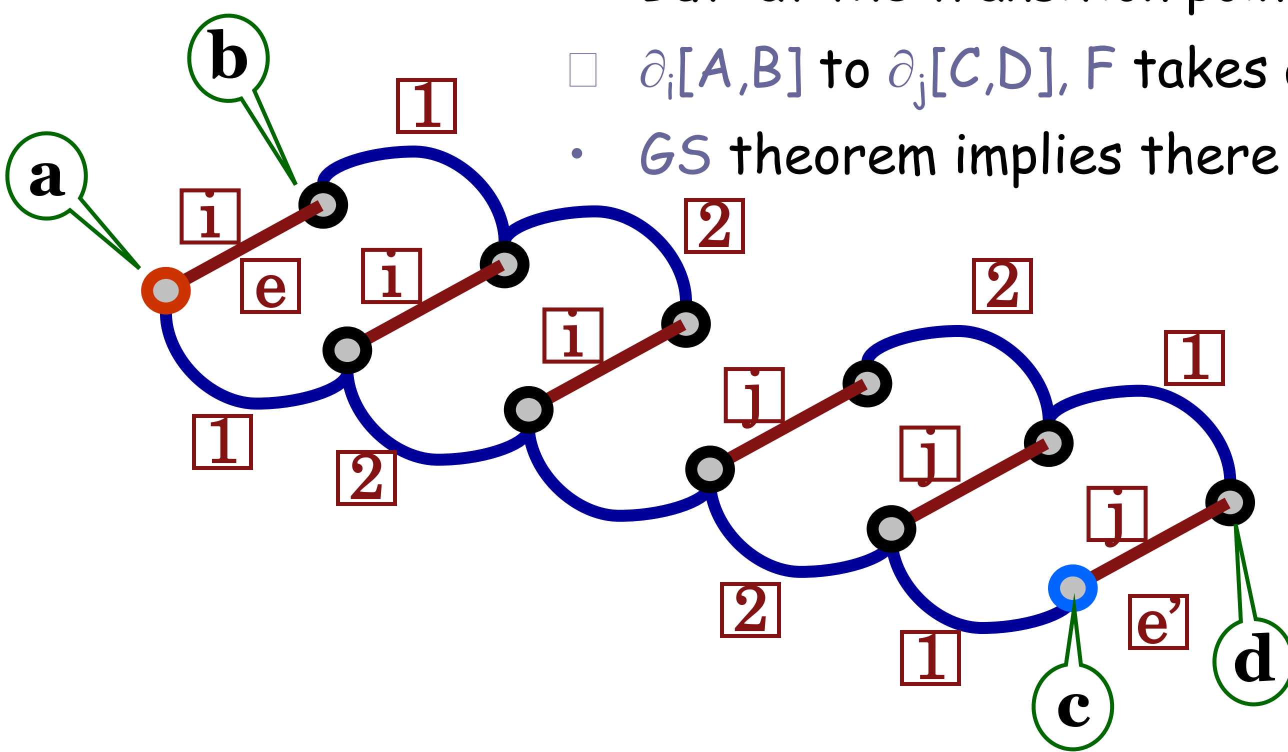


Idea 4: Canonical paths

Define a canonical path $\Gamma\{e,e'\}$ for all $e \in \partial_i[A,B]$ and $e' \in \partial[C,D]$ such that:

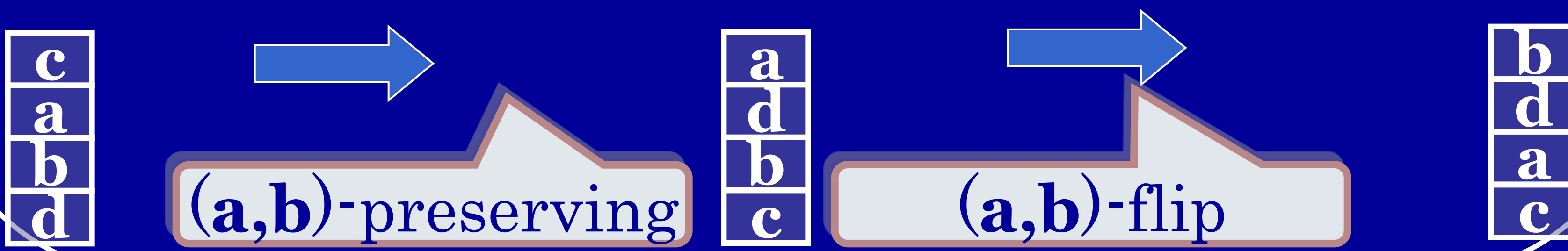
- The path begins at e and ends at e' and
- Path stays in $\partial_i[A,B] \cup \partial_j[C,D]$

  or encounters manipulation
- But: at the transition point m from
- $\partial_i[A,B]$ to $\partial_j[C,D]$, F takes at least 3 values so
- GS theorem implies there exists manipulation.



of Manipulation Points

- $P[M(F))] \geq (4!)^n R^{-1} P[\partial_i[A,B]] \times P[\partial_j[C,D]]$, where
- $R := \max_m \; \#\{ \{e,e'\} : m \text{ is manipulation for } \Gamma\{e,e'\} \}$
- Since: $|M(F)| \geq R^{-1} |\partial_i[A,B]| \times |\partial_j[C,D]|$
- Need to "decode" $\leq$ poly(k,n) $(4!)^n$ (e,e') from m.
- Path to use:
- 1. For all $1 \leq k \leq n$ make k'th coordinate agree with e' except A,B order agrees with e.
- 2. For all $1 \leq k \leq n$ flip (A,B) if need to agree e'.



of Manipulation Points

- Decoding:
- If $e=[x,x']$ and $e'=[y,y']$ suffices to decode (x,y) from m ($(k!)^2$ "pay" to know x' and y').
- Given a hint of size $4n$ know step of the path.
- Suffices for each coordinate s: given m_s decode at most $4!$ Options for (x_s, y_s).
- Given m_s either know x_s, or y_s or $4!/2$ options for x_s and 2 options for y_s.
- Decoding works!

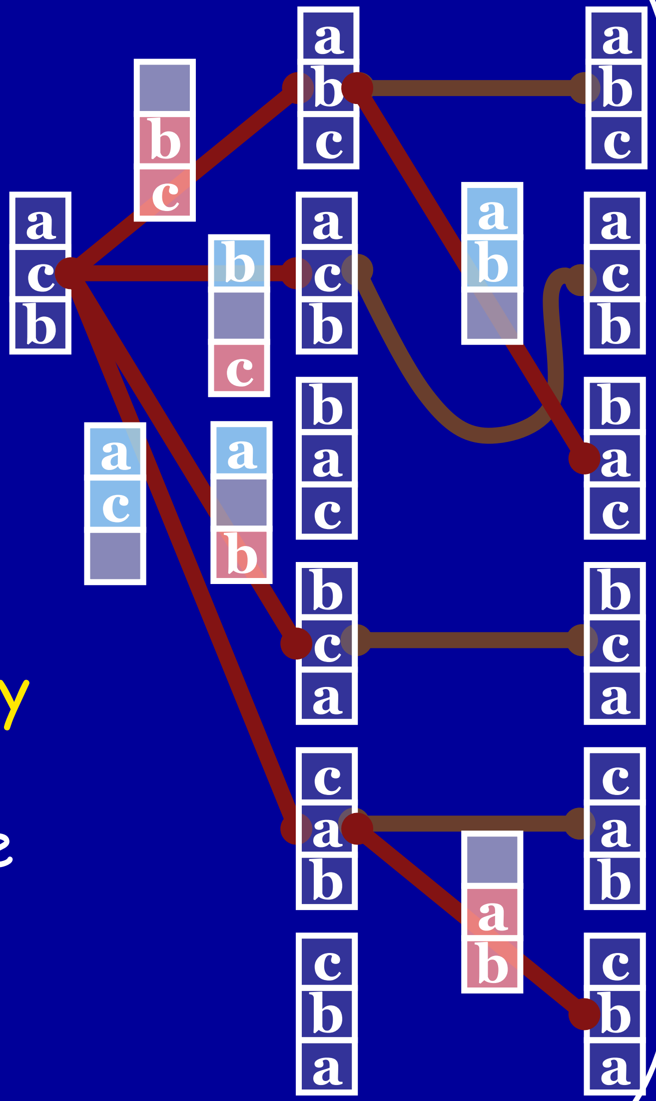
- So $P[M(F))] \geq (4!)^n R^{-1} P[\partial_i(a,b)] \times P[\partial_j(c,d)]$, "gives"
- $P[M(f)] \geq \varepsilon^2 (6n)^{-5}$.
- QED.

However …

- In fact, cheating in various places … - most importantly:
- Manipulation point = x or y up to 3 coordinates, so:
- $R \leq 2\, n\, 4^n\, (k!)^3$
- $P[M(f)] \geq (k!)^{-3}\, \varepsilon^2\, (6n)^{-5}$
- Fine for constant # of alternatives k, but not for large k.

Idea 5: Geometries on the ranking cubes

- To get polynomial dependency on k, use refined geometry:
- (x,x') ∈ Edges if x,x' differ in a single voter and an adjacent transposition.
- For a single voter:
- refined geometry = adjacent transposition card-shuffling.
- Prove: geometry = refined geometry up to poly. factors in k (spectral, isoperimetric quantities behave the same; Aldous-Diaconis, Wilson).
- Prove: Combinatorics still works.
- Gives manipulation by adj. transposition.



Open Problems

- Are there other combinatorial problems where high order interfaces play an interesting role?
- Can other isoperimetric tools be extended to higher order interfaces?
- Tighter results for GS theorem?

- Thank you for your attention!