Social Choice and Social Network

Aggregation of Biased Binary Signals

(Draft)

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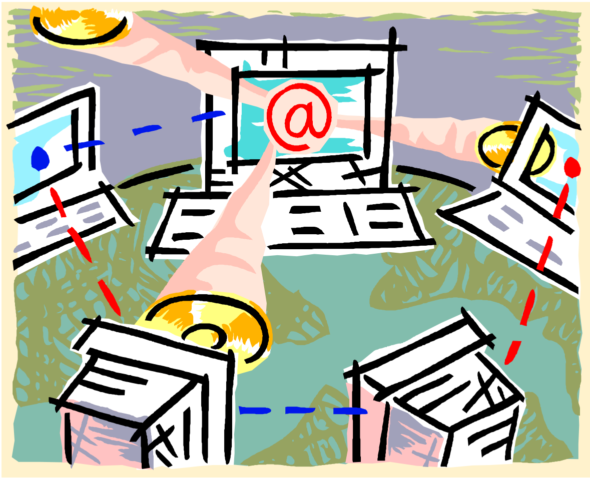

Condorect’s Jury Theorem (1785)

• n juries will take a majority vote between two alternatives - and +.

• Either - or + is *correct*, and each jury votes correctly independently with probability $p > ½$.

• Then as $n \to \infty$:

• correct outcome will be chosen with probability -> 1

• Note: Assume p is fixed (does not depend on n).

• This is referred to as “Aggregation of Information”

Nicola de Condorcet

from wikipidea

- From Wikipedia:

Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet (17.9.1743 – 28.3.1794), known as **Nicolas de Condorcet**, was a French philosopher, mathematician, and early political scientist who devised the concept of a *Condorcet method*. Unlike many of his contemporaries, he advocated a liberal economy, free and equal public education, constitutionalism, and equal rights for women and people of all races. His ideas and writings were said to embody the ideals of the Age of Enlightenment and rationalism, and remain influential to this day. He died a mysterious death in prison after a period of being a fugitive from French Revolutionary authorities.

Condorect’s Jury Theorem (1785)

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Proof of Condorect's Theorem?

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- Recall the law of large numbers

Proof of Condorcet's Jury Theorem

- By law of large numbers
- P[# of voters who vote correctly > 0.5 n] → 1 $\blacksquare$

- (Weak) Law of Large numbers stated by Gerolamo Cardano (1501–1576).
- First proven by Jacob Bernoulli on 1713.

from wikipidea

from wikipidea

Historical Notes

• Cardano is known for solution of some quartic equaltions.
• He was an illegitimate child of a friend of L. Da Vince.
• Survived financially by gambling and playing chess.
• J. Bernoulli from a high standing family.
• Prof. of mathematics at Basel.

from wikipidea

from wikipidea

Proof of Condorcet's Jury Theorem

- By law of large numbers

- P[# of voters who vote correctly > 0.5 n] $\rightarrow$ 1 $\blacksquare$

- Two natural refinements:

- How small can p be as a function of n for the conclusion to hold?

- What is the probability of error for finite n and p?

How small can p be as a function of n for the conclusion to hold?

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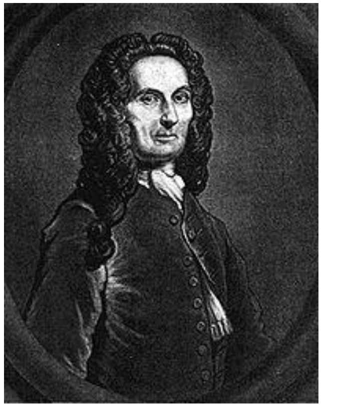
- Recall the Central Limit Theorem.

How small can p be as a function of n for the conclusion to hold?



from wikipidea

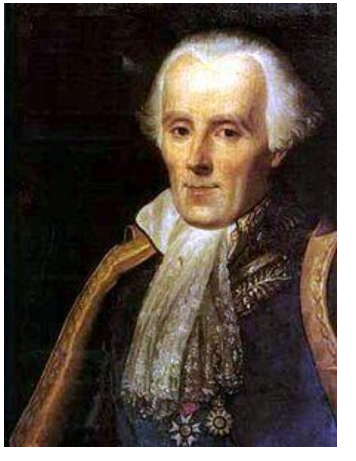
- Let $p(n) = 0.5 + c\, n^{-1/2}$ and
- $q(n)$ = P[Maj is correct] give n ind. $p(n)$ signals
- Then by the CLT

- $\lim q(n) = P(N(0,1^2) > -2c)$

- So if $p-0.5 \gg n^{-1/2}$ then $q(n) \to 1$.

- If $p-0.5 \ll n^{-1/2}$ then $q(n) \to 1/2$

- Explain the conclusion!



from wikipidea

- CLT was established by Moivare on 1733 but was mostly ignored, in particular by Condorcet. Pierre Simone Laplace extended the proof in 1812.

Finite n estimates of correctness

- Large deviations:
- Let $X_1,\ldots,X_n$ be the original signals.
- Let $a = p-0.5$ then
- $P[|\text{Avg } X_i - p| > a] < 2 \exp(-2 a^2 n)$
- So $P[\text{Maj is not correct}] < 2 \exp(-2 a^2 n)$
- Good already when a >> n^{-1/2}
- Classcial large deviation results due to Cramer (beginning of 20^{th} century)

Beyond Condorcet's Jury Theorem

- Further questions:

- What about other aggregation functions?
- E.G: U.S Electoral college?
- Other functions?

- What about non independent signals?
- You and your mom may be (anti) correlated.

Beyond Condorcet's Jury Theorem

- Further questions:

- What about other aggregation functions?
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Beyond Condorcet's Jury Theorem

- First consider

- n independent

- p biased signals

- but other aggregation functions.

The Electoral College example

- Assume $n = m^2$ is an odd square.

- Consider an imaginary country partitioned into m states each with m voters.

- Consider the following voting rule:
- Winner in each state chosen according to majority vote in that state.
- Overall winner = winner in the majority of states.

- Questions:
- Is this method different than majority vote?
- Does the conclusion of the jury theorem still hold?

- To do -> Illustration of function

The Electoral College example

Questions:

- Is this method different than majority vote?
- Yes (to do -> show example)

- Does the conclusion of the jury theorem still hold?
-
- It does – here’s a proof:
- Given $p>1/2$ and m let $q(p,m)$ be the probability that the majority in one of the states is correct. Then $q(p,m) > p > \frac{1}{2}$ and in fact $q(p,m) \to 1$.
- The overall winner is the winner in the majority of states. Thus we have a majority vote with m juries = states and where each state is correct with probability $q(p,m) > p > \frac{1}{2}$.

■

Small Bias in Electoral College

- Assume $n = m^2$ is an odd square.

- What is the smallest bias that guarantees the conclusions of the jury theorem?

-

Small Bias in Electoral College

- Assume $n = m^2$ is an odd square.

- What is the smallest bias that guarantees the conclusions of the jury theorem?

- Claim: Let $p = 0.5 + a/m = 0.5 + a\, n^{-1/2}$ and let
- $p(a)$ = probability outcome is correct as $m \to \infty$.
- Then:
- $p(a)$ is well defined and $p(a) \to 1$ as $a \to \infty$.

- Pf: HW
- Hint: Use the local Central limit theorem.

More examples

• We can similarly try to analyze many more examples.

• HW: Compare Majority and electoral college in the US. What value of p is needed to get the correct outcome with probability 0.9? 0.99?

•Other examples in class?

• However it natural to ask if there are general principals that imply aggregation of information.

• In particular we may want to ask:
What are the best/worst functions for aggregation of information? Are there general conditions that imply aggregation of information?

General functions

- What are the best/worst functions for aggregation of information?

- An aggregation function is just a function $\{-,+\}^n \rightarrow \{-,+\}$

Some bad examples

- What are the best/worst functions for aggregation of information?

- An aggregation function is just a function $\{-,+\}^n \rightarrow \{-,+\}$

- Answer:
- The function that does the opposite of Majority function doesn't aggregate very well …

Monotonicty

- What are the best/worst functions for aggregation of information?

- The function that does the opposite of Majority function doesn't aggregate very well …

- This function is not natural. It is natural to look at monotone functions:

- f is monotone if $\forall$ i $x_i \leq y_i \Rightarrow f(x) \leq f(y)$

- Q: What are the best/worst monotone aggregation functions?

An example

Q: What are the best/worst monotone aggregation functions?

• The constant (monotone) function f = + doesn't aggregate very well either.

Fairness

Q: What are the best/worst monotone aggregation functions?

- The constant (monotone) function f = + doesn't aggregate very well either.

- We want to require that f is fair – treats the two alternatives in the same manner.

- f is fair if f(-x) = -f(x).

- Q: assuming f is monotone and fair what is f(++++++)?
-
- Q: What are the best/worst fair monotone aggregation functions?

Formal definition

Q: What are the best/worst monotone aggregation functions?

- To define the problem more formally assume:
- Apriori correct signal is +/- w.p. ½.
- Each voter receives the correct signal with probability $p > ½$.
- For a fair aggregation function f, let

$$C(p,f) = P[f \text{ results in the correct outcome}] = P[f = + \mid \text{signal} = +]$$

Q: "What are the best/worst fair monotone aggregation functions?" means

Q: What are the fair monotone aggregation functions which minimize/maximize C(p,f)?

The Best Function

Claim: Majority is the best fair monotone symmetric aggregation function (not clear who proved this first – proved in many area independently)

Pf:?

The Best Function

Claim: Majority is the best fair monotone symmetric aggregation function (not clear who proved this first – proved in many area independently)

Pf: $C(f,p) = \sum_x P[x]\, P[f(x) = s \mid x]$

To maximize this over all f need to choose f so that f(x) has the same sign as $(P[s = + \mid x] - P[s = - \mid x])$.

Now by Bayes rule:

$P[s = + \mid x] \,/\, P[s = - \mid x] = P[x \mid s=+] \,/\, P[x \mid s=-] =$

$= a^{\#(+,x)-\#(-,x)}$

where $a = p/(1-p) > 1$ and $\#(+,x)$ is number of +'s in x

So optimal rule chooses $f(x) = \text{sign}(\, n(+,x)-n(-,x))$

The Worst Function

Claim: The worst function is the dictator $f(x) = x_i$.

For the proof we'll use Russo's formula:

Claim 1: If f is a monotone function $f : \{-,+\}^n \to \{-,+\}$ and
$f_i(x) = f(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n) - f(x_1,\ldots,x_{i-1},-,x_{i+1},\ldots,x_n)$

then $C'(f,p) = 0.5 \sum_{i=1}^n E_p[f_i] = \sum_{i=1}^n E_p[Var_{i,p}[f]]/(4p(1-p))$

$Var_i[f] = E_p[\, Var_p[f \mid x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]\,]$

Pf: Use the chain rule and take partial derivatives.

Remark: f_i is closely related to the notion of pivotal voters (economics) and influences in computer science.

The Worst Function

Claim: The worst function is the dictator $f(x) = x_i$.

Remark: This is possibly the first time this is proven so look for bugs!

The second claim we need has to do with local vs. global variances:

Claim 2: $\text{Var}[f] \leq \sum_i \text{Var}_i[f]$ with equality only for functions of one coordinate.

Pf of Claim 2: Possible proofs:

Decomposition of variance of martingales differences

Fourier analysis

The Worst Function

Claim: The worst function is the dictator $f(x) = x_i$.

Claim 1: $C'(f,p) = \sum_{i=1}^n p\, E_p[f_i] = (2(1-p))^{-1} \sum i=1^n n\, Var_i[f]$

Claim 2: $Var[f] \leq \sum_i Var_i[f]$

Pf of main claim:

- For all monotone fair functions we have C(g,0.5)=0.5 and C(g,1)=1.
- Let f be a dictator and assume by contradiction that
- C(f,p) > C(g,p) for some p>1/2.
- Let q = inf {p : C(f,p) > C(g,p)} then
- C(f,q) = C(g,q) and $C'(f,q) \geq C'(g,q)$ so:
- $Var_q[g] = Var_q[f] = \sum_i Var_{i,p}[f] \geq \sum_i Var_{i,p}[g]$
- So g is function of one coordinate.

Other functions?

So far we know that:

1. Majority is the best.
2. Electoral college aggregates well.
3. Dictator is the worst among fair monotone functions and doesn’t aggregate well.
4. What about other functions?

5. Example: Recursive majority (todo: add details and pic)

6. Example: An extensive forum (todo: add details and pic).

The effect of a voter

<u>Def</u>: $E_p[f_i]$ is called the influence of voter i.

<u>Theorem (Talagrand 94):</u>

- Let f be a monotone function.
- If $\delta = \max_x \max_i E_x[f_i]$ and $p < q$ then
- $E_p[f \mid s = +] \, (1-E_q[f \mid s=+]) \leq \exp(c \ln \delta \, (q-p))$
- for some fixed constant c>0.

- In particular: if f is fair and monotone, taking p=0.5:

- $E_q[f \text{ is correct}] \geq 1- \exp(c \ln \delta \, (q-0.5))$

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- In particular: if f is fair and monotone, taking p=0.5:

- $E_q[f \text{ is correct}] \geq 1- \exp(c \ln \delta (q-0.5))$

- This means that if each voter has a small influence then the function aggregates well!

An important case

<u>Def:</u> A function $f: \{-,+\}^n \rightarrow \{-,+\}$ is <u>transitive</u> if there exists a
- group G acting transitively on [n] s.t.
- for every $x \in \{-,+\}^n$ and any $\sigma \in G$ it holds that $f(x_\sigma) = f(x)$,

where
- $x_\sigma(i) = x(\sigma(i))$

<u>Thm (Friedgut-Kalai-96) :</u>
- If f is transitive and monotone and
- $E_p[f \mid s= +] > \varepsilon$ then
- $E_q[f \mid s = +] > 1-\varepsilon$ for $q=p+c \log(1/2\varepsilon)/ \log n$

<u>Note:</u> If f is fair transitive and monotone
we obtain
E_q[f is correct] $> 1-\varepsilon$ for $q=0.5+c \log(1/2\varepsilon)/ \log n$

An important case

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- If f is transitive and monotone and
- $E_p[f] > \varepsilon$ then
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- Note: If f is fair transitive and monotone we obtain $E_q[f \text{ is correct}] > 1-\varepsilon$ for $q=0.5+c \log(1/2\varepsilon)/ \log n$

- This implies aggregation of information as long as the signals have correlation at least $0.5+c/\log n$ with the true state of the world.

Examples of aggregation / no aggregation

Claim:

Examples: Electoral college

Example: Recursive Majority

Example: Hex Vote

Note: The results actually hold as long as there are finitely many types all of linear size in n.

Other distributions

So far we have discussed situations were signals were independent. What is signals are dependent?

Setup: Each voter receives the correct signal with probability p
But: signals may be dependent.

Question: Does Condorcet Jury theorem still hold?

Other distributions

So far we have discussed situations were signals were independent. What is signals are dependent?

Setup: Each voter receives the correct signal with probability p
But: signals may be dependent.

Question: Does Condorcet Jury theorem still hold?

A: No. Assume:

1. With probability 0.9 all voters receive the correct signal.
2. With probability 0.1 all voters receive the incorrect signal.

Other distributions

This example is a little unnatural. Note that in this case just looking at one voter we know the outcome of the election.

Def: The effect of voter i on function $f: \{0,1\}^n \rightarrow \{0,1\}$ for a probability distribution P is:

$e_i(f,P) = E[f \mid X_i = 1] – E[f \mid X_i = 0]$.

Note: Assume $E[X_i] = p$ then:
$Cov[f,X_i] = E[f(X_i – p)] =$
$p\, E[(1-p)\, f \mid X_i = 1] + (1-p)\, E[-p\, f \mid X_i = -1] = p(1-p)\, e_i(f,P)$

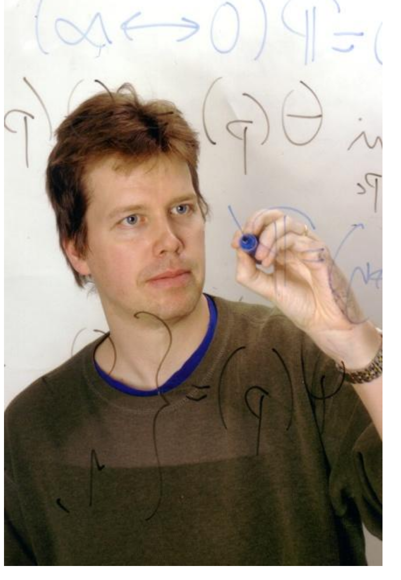
Condorcet's theorem for small effect functions

Theorem (Haggstrom, Kalai, Mossel 04):

•Assume n individuals receive a 1,0 signal so that $P[X_i = 1] \geq p > ½$ for all i.

• Let f be the majority function and assume $e_i(f,P) \leq e$ for all i.

• Then the probability that majority will aggregate correctly is at least: $1 – e/(p-0.5)$.



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• Then the probability that majority will aggregate correctly is at least: $1 – e/(p-0.5)$.

• Proof: Let $Y_i = p_i - X_i$ and $g= 1-f$. Then

• $E[(\sum Y_i) g] = E[g] E[\sum Y_i \mid f = 0] \geq n (p-1/2) E[g]$

•$E[(\sum Y_i) g] = \sum E[Y_i g] = \sum Cov[X_i,f] = \sum p_i(1-p_i) e_i(f) \leq n p(1-p) e$

So $n(p-1/2)E[g] \leq n p (1-p) e \Rightarrow E[g] \leq ep(1-p)/(p-0.5)$
And $E[f] \geq 1 – ep(1-p)/(p-0.5)$.

Comments about the proof

- Proof actually works for all weighted majority functions.

- So for weighted majority functions we have aggregation of information as long as they have small effects.

- In fact the following is true:

Theorem (HKM-04)

- If f is transitive, monotone and fair and is not simple majority then there exists a probability distribution so that: $E[X_i] > ½$ for all i and $E[f] = 0$ and $e_i(f,P) = 0$ for all i.

- If f is monotone and fair and is not simple majority then there exists a probability distribution so that: $E[X_i] > ½$ for all i and $E[f] = 0$ and $e_i(f,P) = 0$ for all i.

Comments about the proof

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$E[X_i] > ½$ for all i and $E[f] = 0$ and $e_i(f,P) = 0$ for all i.

Open Problems in the area

- The general open problem is to understand conditions on distributions of votes and functions which imply aggregation of information.

- Natural conditions include monotonicity of the function, of the measure etc.

- At the practical level it is hard to “check” if a certain voting system has small effects or not.

HW

- The HW is due in 2 weeks.
-
- Please work in groups of 2-4 students preferably from different departments.
-
- Each student should submit her own hw.
-
- Please write your name, student i.d. and the names and i.d.'s of your group members.

HW1

1 Suppose $X_1,\ldots,X_n$ are ind. Signals which are correct with probabilities $p_1,\ldots,p_n$. And $Y_1,\ldots,Y_n$ are ind. Signals which are correct with probability $q_1,\ldots,q_n$.

-
- Assume that f is monotone and fair and that it returns the correct signal for the X's with probability at least $1-\delta$. Show that the same is true for the Y's if $q_i \geq p_i$ for all i.

- In words – if f aggregates well under some signals it aggregates even better under a stronger signal.

HW2

- Consider the electoral college example with m states of size m each where m is odd.

- Show that a signal of strength 0.5 + 1000/m results in an aggregation function which returns the correct result with probability at least 0.99 for all m sufficiently large.

- Hint: Use the local central limit theorem.

HW3

- Compare the actual electoral college used in the US in the last elections to a simple majority vote in terms of the quality of independent signals needed to return the correct result with probability 0.9 and 0.99.

HW4

- Give a complete proof that

- If f is transitive, monotone and fair and is not simple majority then there exists a probability distribution so that:
$E[X_i] > ½$ for all i and $E[f] = 0$ and $e_i(f,P) = 0$ for all i.

- Construct such P for the $m \times m$ electoral college.

HW 5

- What kind of data can give estimates on the effects of voters in real voting systems?

HW 6 – Bonus Problem

- Use the chain rule to prove Russo's formula.

- Let $f : \{-,+\}^n \rightarrow \{-,+\}$. Consider i.i.d. $X_1,\ldots,X_n$ such that $P[X_i = +] = p$. Show that $Var[f] \leq \sum Var_i[f]$.

- Hint: Use Fourier Analysis to express both $Var[f]$ and $Var_i[f]$