Lecture 3

Lecture date: Sep 6, 2005 *Scribe: Josephine Yu*

Let $(\Omega_j, \mathcal{F}_j, \mu_j)$ be a separable probability space for each $j \in \{1, \ldots, m\}$, and let $U^j = \{u_0^j, u_1^j, \ldots\}$ be a standard basis of $L^2(\mu_j)$. Then the elements $u_I = u_{i_1}^1 \otimes \cdots \otimes u_{i_m}^m$ for $I = \{i_1, \ldots, i_m\} \subset \mathbb{N}^m$ form a standard basis for $L^2(\Pi\mu_j)$.

Definition 1 *For $S \subseteq \{1, \ldots, m\}$, let*

$$L_S^2 = span(u_I : i_n \neq 0 \iff n \in S), and$$
$$L_{|S}^2 = functions\ depending\ only\ on\ the\ coordinates\ in\ S.$$

Lemma 2 (From last lecture.) *For any $S \subset [m]$, we have*

$$L_{|S}^2(\Pi\mu_j) = \bigoplus_{T \subseteq S} L_T^2(\Pi\mu_j).$$

Corollary 3 *The spaces L_S^2 do not depend on the (standard) basis.*

Proof: Proof by Induction on $|S|$: If $|S| = 0$, i.e., $S = \emptyset$, then $L_\emptyset^2 = span(u_0^1 \otimes \cdots \otimes u_0^m) = span(1)$, so it is independent of the basis. If $|S| > 0$, then $L_{|S}^2 = L_S \oplus \bigoplus_{T \subsetneq S} L_T$. Since $L_{|S}^2$ does not depend on the basis (by definition), and L_T^2 does not depend on the basis for $T \subsetneq S$ by the inductive hypothesis, L_S^2 does not depend on the basis. (Recall the linear algebra fact that $A \oplus B = A \oplus B' \implies B = B'$.) $\Box$

For $f \in L^2(\Pi\mu_j)$, denote $\hat{f}(I) := \langle f, u_I \rangle$. So $f = \sum_I \hat{f}(I) u_I$. Also, $f = \sum_{S \subseteq [m]} f_S$ where f_S is the projection of f onto L_S^2.

Lemma 4

$$E[f] = \hat{f}(0, \ldots, 0) = f_\emptyset,$$
$$cov(f, g) = \sum_{I \neq (0, \ldots, 0)} \hat{f}(I)\hat{g}(I) = \sum_{S \neq \emptyset} \langle f_S, g_S \rangle$$

Question 5 (Asked by Omid.) *What is L_S for $\{-1, 1\}_0^n$?*

Answer 6 *The only standard basis for $\{-1,1\}_0$ is $\{1,x\}$. We can prove by induction that*

$$L_S = span \prod_{i\in S} x_i.$$

Example 7 *Let $\omega = \{-1,1\}_0^n$. Let f be the majority function, i.e., $f(\mathbf{x}) = maj(x_1,\ldots,x_n) = sgn(x_1+\cdots+x_n)$. Then f is antisymmetric, i.e., $f(\mathbf{x}) = \frac{f(\mathbf{x})-f(-\mathbf{x})}{2}$, so $\hat{f}(S) = 0$ if $|S|$ is even. And f is transitive, i.e., $f(x) = f(\sigma(x))$ where σ is any permutation in S_n, so $\hat{f}(S)$ only depends on $|S|$.*

If $n = 3$, then f has an expansion $f(\mathbf{x}) = \frac{1}{2}(x_1 + x_2 + x_3 - x_1x_2x_3)$.

Example 8 (Projective Geometry Example.)

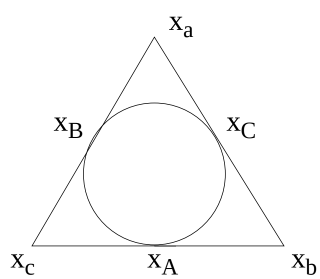

Figure 1: "Projective plane" for Example 8

Let $f : \{-1,1\}^6 \to \{-1,1\}$ be given by $f(\mathbf{x}) = Proj(x) = \sum$ "lines" $= x_ax_bx_C + x_ax_Bx_c + x_Ax_bx_c - x_Ax_Bx_C)/2$. The third monomial is always equal to the product of the first three, so it cancels with one of the first three and the remaining two terms have the same sign. Hence the only possible values of f are ± 1.

Exercise 9 *Prove that in order to fix the function, you have to set at least 5 variables.*

Example 10 (Compositions.) *We know the majority function on three variables, so we can compute*

$$\begin{aligned}
f(\mathbf{x}) =& maj(x_1x_2, maj(x_3,x_1,x_4), maj(x_4,x_5,x_1)) \\
=& \frac{1}{2}(x_1x_2 + maj(x_3,x_1,x_4) + maj(x_4,x_5,x_1)) - \frac{1}{2}x_1x_2\ maj(x_3,x_1,x_4)maj(x_4,x_5,x_1) \\
=& \frac{1}{2}(x_1x_2 + \frac{1}{2}(x_3+x_1+x_4-x_3x_1x_4) + \frac{1}{2}(x_4+x_5+x_1-x_4x_5x_1) \\
&- \frac{1}{2}x_1x_2\frac{1}{2}(x_3+x_1+x_4-x_3x_1x_4)\frac{1}{2}(x_4+x_5+x_1-x_4x_5x_1)
\end{aligned}$$

Exercise 11 *Find a function* $f : \{-1,1\}^{36} \to \{-1,1\}$ *with deg*$(f) = 9$ *such that you need to set at least 25 variables in order to fix it.*

Exercise 12 *Find* $f : \{-1,1\}^n \to \{-1,1\}$ *for arbitrarily large* n *of degree* $\frac{n}{4}$ *such that you need to set at least* $\frac{25}{36}n$ *variables in order to fix it.*

(The last three exercises together worth 5 points; The example is due to Oded Regev)

Hermite Expansion

Let γ be the one dimensional Gaussian measure, with density $\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$, and

$$L^2(\gamma) = \{f : \mathbb{R} \to \mathbb{R} \mid \int_{\mathbb{R}} |f(x)|^2 d\gamma(x) < \infty\}.$$

Definition 13 *The Hermit polynomials are*

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{\frac{-x^2}{2}},$$

and the normalized Hermit polynomials are

$$h_n = \frac{1}{\sqrt{n!}} H_n.$$

Lemma 14 *The function* H_n *(hence* h_n*) is a polynomial of degree* n*.*

Proof: Expand the derivative and note that the exponentials cancel and that the highest degree (n) term does not cancel. □

Lemma 15 *The polynomials* H_n *(hence* h_n*) form a linear basis of* $L^2(\Gamma)$*.*

Proof: From the previous lemma, H_n's form a basis for the space of all polynomials, and the polynomials are dense in $L^2(\gamma)$. □

Lemma 16 *The normalized Hermit polynomials* $\{h_n\}$ *form a standard basis for* $L^2(\gamma)$*.*

We need the following lemma for the proof.

Lemma 17 *The Hermit polynomials can be written explicitly as*

$$H_n(x) = x^n - \frac{n(n-1)}{1!2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!2^2}x^{n-4} + \dots.$$

Proof: This formula gives $H_0 = 1$. Suppose the lemma holds for $n \geq 0$. Then

$$\begin{aligned} H_{n+1}(x) &= (-1)^{n+1} e^{\frac{x^2}{2}} \frac{d^{n+1}}{dx^{n+1}} e^{\frac{-x^2}{2}} \\ &= (-1) e^{\frac{x^2}{2}} \frac{d}{dx}(H_n(x) e^{\frac{-x^2}{2}}) \\ &= x H_n(x) - \frac{d}{dx} H_n(x). \end{aligned}$$

Substituting the formula for H_n yields the correct formula for H_{n+1}. □

By differentiating the RHS in the previous lemma, we get

$$\frac{d}{dx} H_n(x) = n H_{n-1}(x).$$

Proof:[Lemma 16] We already know that $\{h_n\}$ is a linear basis. By definition, $h_0 = 1$. We need to check orthogonality. First check that

$$\langle H_n, H_0 \rangle = \frac{(-1)^n}{\sqrt{2\pi}} \int \frac{d^n}{dx^n} e^{\frac{-x^2}{2}} dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}.$$

Now suppose $m \geq n > 0$. Using integration by parts and induction,

$$\begin{aligned} \langle H_n, H_m \rangle &= \int H_n(x) H_m(x) d\gamma(x) \\ &= \frac{(-1)^m}{\sqrt{2\pi}} \int H_n(x) \frac{d^m}{dx^m} e^{\frac{-x^2}{2}} dx \\ &= \frac{(-1)^{m-1}}{\sqrt{2\pi}} \int \frac{d}{dx} H_n(x) \frac{d^{m-1}}{dx^{m-1}} e^{\frac{-x^2}{2}} dx \\ &= \frac{(-1)^{m-1} n}{\sqrt{2\pi}} \int H_{n-1}(x) \frac{d^{m-1}}{dx^{m-1}} e^{\frac{-x^2}{2}} dx \\ &= n \langle H_{n-1}, H_{m-1} \rangle \\ &= \begin{cases} n! & \text{if } n = m \\ 0 & \text{if } m > n \end{cases} \text{ by induction.} \end{aligned}$$

□

Exercise 18 (2 points.) *Prove that*

$$H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^k H_{n-k}(y),$$

$$E_{x \to \gamma}[H_n(x+a)] = a^n \textit{ for any constant } a \in \mathbb{R}.$$

Theorem 19 *Let* $f \in L^2(\gamma)$ *with* $\gamma(\{x : f \textit{ is not continuous at } x\}) = 0$. *Consider* $\{-1,1\}^n_\theta$, *and let* $f_n(x_1, \ldots, x_n) = f\left(\frac{\sum_{i=1}^n (x_i - \theta)}{\sqrt{n(1-\theta^2)}}\right)$. *Let* $f = \sum_{k=0}^\infty \hat{f}(k) h_k$, *and* $f_n = \sum_{k=0}^n \hat{f}_n w_k^n$ *where* $w_k^n = \frac{1}{z_k} \sum_{S:|S|=k} \prod_{i \in S}(x_i - \theta) \in L^2(\{-1,1\}^n_\theta)$ *and* z_k *is such that* $|w_k^n| = 1$.

Then, for every k, *it holds that*

$$\lim_{n \to \infty} \hat{f}_n(k) = \hat{f}(k).$$

Next time, we will use this theorem to get an asymptotic expansion of the majority function.