Lecture 12

Lecture date: October 6, 2005 *Scribe: Lionel Levine*

Recall that an operator $T : L^p(\Omega, \mu) \to L^q(\Omega, \mu)$ is (p, q)-hypercontractive $(1 \leq p < q)$ if

$$||Tf||_q \leq ||f||_p$$

for all $f \in L^p(\Omega, \mu)$.

Proposition 1 *Let $T_i : L^p(\Omega_i, \mu_i) \to L^q(\Omega_i, \mu_i)$, $i = 1, 2$. If T_1 and T_2 are (p, q)-hypercontractive, so is $T_1 \otimes T_2$.*

The proof uses the following fact, which is known as the generalized Minkowski inequality.

Exercise 2 *(1 pt) Given $f : (\Omega_1, \mu_1) \times (\Omega_2, \mu_2) \to \mathcal{R}$ and $x \in \Omega_1$, let $||f||_{x,p} = ||g_x||_p$, where $g_x(y) = f(x, y)$; let $||f||_{y,p}$ be defined analogously for $y \in \Omega_2$. Prove that if $1 \leq p \leq q$*

$$||\ ||f||_{x,p}\ ||_q \leq ||\ ||f||_{y,q}\ ||_p.$$

Proof:[Proof of Proposition 1] Let $T = T_1 \otimes T_2$. Then $T = T_1^* T_2^*$, where $T_1^* = T_1 \otimes 1$, $T_2^* = 1 \otimes T_2$. By Fubini's theorem and the generalized Minkowski inequality

$$\begin{aligned} ||Tf||_q &= ||\ ||T_1^*(T_2^* f)||_{x,q}\ ||_q \\ &\leq ||\ ||T_2^* f||_{x,p}\ ||_q \\ &\leq ||\ ||T_2^* f||_{y,q}\ ||_p \\ &\leq ||\ ||f||_{y,p}\ ||_p \\ &= ||f||_p. \end{aligned}$$

□

Recall that the *Bonami-Beckner operator* T_η on $L^2(\{1, -1\}_0)$ is given by

$$T_\eta(f) = \eta f + (1 - \eta) Ef.$$

Here η is a parameter taking values in $[0, 1]$. The aim of this lecture is to prove the following theorem of Bonami and Beckner.

Theorem 3 *T_η is (p, q)-hypercontractive if $\eta^2 \leq \frac{p-1}{q-1}$.*

We are really interested in tensor products $T_\eta \otimes \ldots \otimes T_\eta$ acting on $L^2(\{1,-1\}_0^n)$. According to Proposition 1, the tensor product is (p,q)-hypercontractive whenever T_η is.

Exercise 4 *(1 pts) Show that the bound on η in the theorem is tight, i.e. the converse holds.*

For $p > 1$ let p' be the unique solution to $\frac{1}{p} + \frac{1}{p'} = 1$. Recall that in Holder's inequality

$$||fg||_1 \leq ||f||_p ||g||_{p'}$$

equality is attained when $g = |f|^{p/p'}$. Hence

$$||f||_p = \sup\{||fg||_1 \ : \ ||g||_{p'} = 1\}. \tag{1}$$

Lemma 5 *It suffices to prove Theorem 3 under the following assumptions.*

(1) $\eta^2 = \frac{p-1}{q-1}$;

(2) $1 < p < q < 2$.

Proof: If $\eta^2 < \frac{p-1}{q-1}$, choose $p^* < p$ for which $\eta^2 = \frac{p^*-1}{q-1}$. If the theorem holds with η satisfying (1), then by the monotonicity of norms we have

$$||T_\eta f||_q \leq ||f||_{p^*} \leq ||f||_p,$$

so it holds in general.

For condition (2), note that by continuity, if the theorem holds for $1 < p < q < 2$, then it holds for $1 \leq p < q \leq 2$. There are two remaining cases.

Case 1: $2 < p < q$. Then $1 < q' < p' < 2$, and since $(p-1)(p'-1) = 1 = (q-1)(q'-1)$ we have $\eta^2 = \frac{p-1}{q-1} = \frac{q'-1}{p'-1}$. Thus we may assume that T_η is (q', p')-hypercontractive. By (1) and the self-adjoint property ("reversibility") of T_η, we have

$$\begin{aligned} ||T_\eta f||_q &= \sup\{||gT_\eta f||_1 \ : \ ||g||_{q'} = 1\} \\ &= \sup\{||fT_\eta g||_1 \ : \ ||g||_{q'} = 1\} \\ &\leq ||f||_p \sup\{||T_\eta g||_{p'} \ : \ ||g||_{q'} = 1\} \\ &\leq ||f||_p \end{aligned}$$

where in the last step we have used the fact that $||T_\eta g||_{p'} \leq ||g||_{q'}$.

Case 2: $p < 2 < q$. We will use the "semigroup property" of the Bonami-Beckner operators, i.e. the fact that $T_{\eta_1 \eta_2} = T_{\eta_1} T_{\eta_2}$. Write $\eta = \eta_1 \eta_2$, where $\eta_1^2 = p - 1$, $\eta_2^2 = \frac{1}{q-1}$. By case 1 we may assume T_{η_1} is $(2,q)$-hypercontractive, and since $p<2$ we may assume T_{η_2} is $(p,2)$-hypercontractive, hence

$$||T_\eta f||_q = ||T_{\eta_2} T_{\eta_1} f||_q \leq ||T_{\eta_1} f||_2 \leq ||f||_p.$$

□

Proof:[Proof of Theorem 3] Since $||\ T_\eta|g|\ ||_q \geq ||T_\eta g||_q$ our test function g can be taken nonnegative. Moreover if $g \geq 0$ and $g \not\equiv 0$ then $T_{1-\epsilon} g > 0$ for any $\epsilon > 0$. Since $T_\eta = T_{\frac{\eta}{1-\epsilon}} T_{1-\epsilon}$, by continuity in η we may assume $g > 0$. After scaling by a positive constant factor, g has the form $g(x) = 1 + ax$ with $|a| < 1$. Thus $T_\eta g(x) = 1 + a\eta x$ and

$$\begin{aligned} ||T_\eta g||_q^q &= \frac{1}{2}(1+a\eta)^q + \frac{1}{2}(1-a\eta)^q \\ &= \sum_{n\geq 0} \binom{q}{2n} a^{2n}\eta^{2n}. \end{aligned}$$

Using the fact that $(1+x)^{p/q} \leq 1 + \frac{p}{q}x$ for $x \geq -1$, we obtain

$$||T_\eta g||_q^p = 1 + \frac{p}{q}\sum_{n\geq 1} \binom{q}{2n} a^{2n}\eta^{2n}.$$

Since $||g||_p^p = \sum_{n\geq 0} \binom{p}{2n} a^{2n}$ it suffices to show $\binom{p}{2n} \geq \frac{p}{q}\binom{q}{2n}\eta^{2n}$ for all $n \geq 1$. Indeed, recalling $\eta^2 = \frac{p-1}{q-1}$ we obtain

$$\begin{aligned} \binom{p}{2n}^{-1} \frac{p}{q} \binom{q}{2n} \eta^{2n} &= \frac{(q-1)(q-2)\dots(q-2n+1)}{(p-1)(p-2)\dots(p-2n+1)} \left(\frac{p-1}{q-1}\right)^n \\ &= \left(\frac{p-1}{q-1}\right)^{n-1} \prod_{m=2}^{2n-1} \left(\frac{m-q}{m-p}\right) \\ &\leq 1 \end{aligned}$$

where in the last step we have used the assumption $1 < p < q < 2$. □

This takes care of the space $\{1,-1\}_0$. The following theorem of Oleszkiewicz gives the $(2,q)$-constant of hypercontractivity for the more general spaces $\{-1,1\}_\theta$.

Theorem 6 *If $\mu(-1) = \alpha$, $\mu(1) = \beta$ with $\alpha < \beta$, then T_η is $(2,q)$-hypercontractive if*

$$\eta^2 \leq \sigma(2,q) := \frac{\beta^{2/q} - \alpha^{2/q}}{\alpha^{2/q-1}\beta - \beta^{2/q-1}\alpha}. \tag{2}$$

and $(p,2)$-hypercontractive if

$$\eta^2 \leq \sigma(p,2) := \frac{\beta^{2-2/p} - \alpha^{2-2/p}}{\beta\alpha^{1-2/p} - \alpha\beta^{1-2/p}} \tag{3}$$

Exercise 7 *(10 pts + final project) In the setting of the above theorem, find the (p,q) constant of hypercontractivity. (This is an open problem.)*

Exercise 8 *(1 pt) In the above theorem, show how to get* $\sigma(2, q)$ *from* $\sigma(p, 2)$ *and vice versa.*

Exercise 9 *(4 pts) Prove either (3) or (2).*

Exercise 10 *(8 pts + final project) Find a short (<1 page) proof of either (3) or (2).*