Lecture 12

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1 Noise Operators

Definition 1 *An operator* $T : L^2(\mu) \to L^2(\mu)$ *is* positivity improving *if* $Tf \geq 0$ *whenever* $f \geq 0$.

Definition 2 *An operator* $T : L^2(\mu) \to L^2(\mu)$ *is called* noise operator *if the following four conditions hold:*

1. T *is positivity improving;*
2. $T(1) = 1$ *(1 is constant function 1);*
3. $|Tf|_2 \leq |f|_2$;
4. $< f, Tg >=< Tf, g >$ *(generalization of Markov reversibility).*

Example 3 *Let* (Ω, μ) *be a finite probability space. Let* M *be a reversible finite Markov chain (MC). Represent* M *by a* $|\Omega| \times |\Omega|$ *transition matrix. Define*

$$T_M f(\cdot) = \sum_y M(\cdot, y) f(y).$$

We claim that T_M *is a noise operator. Properties 1 and 2 follow from the fact that* M *is a transition probabilities matrix. To show 3 and 4 we'll need to evoke M's reversability:*

$$\begin{aligned} |Tf|_2^2 = \sum_x \mu(x)(Tf(x))^2 &= \sum_x \mu(x)(\sum_y M(x,y)f(y))^2 \leq \\ \sum_{x,y} \mu(x)M(x,y)f(y)^2 &= \sum_y \mu(y)f(y)^2 \sum_x M(y,x) = |f|_2^2 \end{aligned} \quad (1)$$

To see condition 4, expand:

$$\begin{aligned} < T_M f, g >= \sum_x \mu(x)Tf(x)g(x) &= \sum_{x,y} \mu(x)M(x,y)f(y)g(x) = \\ \sum_{x,y} \mu(y)M(y,x)f(y)g(x) &= < f, Tg > \end{aligned} \quad (2)$$

Example 4 *Let γ_n be an n-dimensional Gaussian measure. For $\rho \in [0,1]$ define Orenstein-Uhlenbeck operator $T_\rho : L^2(\gamma_n)) \to L^2(\gamma_n)$ as*

$$(T_\rho f)(x) = \mathbf{E}_{y\sim\gamma_n}[f(\rho x + \sqrt{1-\rho^2}y)]$$

We claim that operator T_ρ is a noise operator.

Proof: *Conditions 1 and 2 follow immediately from definition. Check condition 3:*

$$\begin{aligned} |T_\rho f|_2^2 = \mathbf{E}_{x\sim\gamma_n}(T_\rho f(x)^2) &= \mathbf{E}_{x\sim\gamma_n}(\mathbf{E}_{y\sim\gamma_n} f(\rho x + \sqrt{1-\rho^2}y)^2) \leq \\ \mathbf{E}_{x,y\sim\gamma_n}(f(\rho x + \sqrt{1-\rho^2}y)^2) &= \mathbf{E}_{x\sim\gamma_n}(f(x)^2) = |f|_2^2 \end{aligned} \quad (3)$$

Reversability condition 4:

$$< T_\rho f, g >= \mathbf{E}_{x,y\sim\gamma_n}[f((\rho x + \sqrt{1-\rho^2}y))g(y)] \quad (4)$$

Let $Z = (\rho x + \sqrt{1-\rho^2}y)$, $W = y$ where x and y distributed as above. Then $Z, W \sim \gamma_n$ and also are correlated: $\mathbf{E}Z_iW_j = \rho\delta_{ij}$. But definition of Z, W is invariant under exchange of X, Y we have

$$< T_\rho f, g >= \mathbf{E}[f(Z)g(W)] =< f, T_\rho g >$$

. $\square$

2 Tensoring

Definition 5 *For $i = 1, \ldots, n$ let $T_i : L^2(\mu_i) \to L^2(\mu_i)$ be noise operators. Let $T = \bigotimes T_i : L_2(\prod_{i=1}^n \mu_i) \to L_2(\prod_{i=1}^n \mu_i)$ be a new operator satisfying*

$$T(\otimes_i u_i) = \otimes_i(T_i u_i)$$

for all $u_i \in L^2(\mu_i)$.

Proposition 6 *Operator T is well-defined.*

Proof: To show that the operator is not overly defined, pick two separate basis $\mathcal{U}^1 \otimes \ldots \otimes \mathcal{U}^n$, $\mathcal{V}^1 \otimes \ldots \otimes \mathcal{V}^n$ and extend (multi-linearly) operators from values on basis elements. It's sufficient to consider only a pair of basis of this form: $\mathcal{U}^1 \otimes \mathcal{U}^2 \otimes \ldots \otimes \mathcal{U}^n$, $\mathcal{V}^1 \otimes \mathcal{U}^2 \otimes \ldots \otimes \mathcal{U}^n$. However for this case equality of operators follows from linearity of T_1. $\square$

Definition 7 *Given T_i (as above), define new operators $T_i^* : L_2(\prod_{i=1}^n \mu_i) \to L_2(\prod_{i=1}^n \mu_i)$ as*

$$(T_i^*(f(\cdot, \ldots, \cdot)))(x_1, \ldots, x_n) = (T_i(f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)))(x_i)$$

Proposition 8 *$\bigotimes_i T_i = \prod_i T_i^*$ and all T_i's commute.*

Proof: Apply T_i^* to $f = \otimes u_i$:

$$T_i^*(u_1 \otimes \ldots \otimes u_n) = u_1 \otimes \ldots \otimes u_{i-1} \otimes T_i u_i \otimes u_{i+1} \otimes \ldots \otimes u_n$$

□

Remark 9 *If $T_i = T_{M_i}$, then $\bigotimes_i T_i = T_M$ where $M(x,y) = \prod_{i=1}^n M(x_i, y_i)$*

Definition 10 *For $\rho \in [0,1]$ define Bonami-Beckner operator $T_\rho : L_2(\prod_{i=1}^n \mu_i) \to L_2(\prod_{i=1}^n \mu_i)$ as $T_\rho = \bigotimes T_i$ where $T_i(f_i) = \rho f_i + (1-\rho)\mathbf{E}_{\mu_i}[f_i]$*

We can think of T_i as $T_i f(x) = \mathbf{E} f(y)$ where y is a ρ-correlated copy of x and all coordinates are treated independently.

Remark 11 *For a finite space $(\Omega \mu)$, $T_i = T_{M_i}$, where $M_i(x,y) = \rho \delta_{y=x} + (1-\rho)\mu$*

Proposition 12 *Bonami-Beckner operator is a noise operator.*

Proof: Proof analogous to the same proof for Orenstein-Uhlenbeck operator if we use the form $Tf(x) = \mathbf{E}_{y:\mathbf{E}xy=\rho}(f(y))$. □

3 Hypercontractivity

Exercise 13 *Show that if M is a MC, then $|T_M f|_p \leq |f|_p$ for all $p \geq 1$.*

Definition 14 *Let $1 \leq p \leq q$. Operator T is $(p,q)-$hypercontractive if*

$$|Tf|_q \leq |f|_p$$

for all f such that $|f|_p < \infty$

Next time we will show that if all T_i are (p,q)-hypercontractive then $\bigotimes_i T_i(p,q)$ is (p,q)-hyper-contractive as well.