Lecture 17

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Our main result in this lecture is a proof that a function with low influence sum is "simple"—i.e. it depends on a small number of variables. This uses hypercontractivity results from previous lectures.

1 Low-Influence-Sum Functions Depend on Few Coordinates

The following theorem generalizes Friedgut [Fr98].

Theorem 1 *Let μ be a probability distribution where the smallest atom has probability α. Let $f \in L^2(\mu^n)$ be ± 1-valued with*

$$I(f) = \sum_{i=1}^{n} I_i(f) = \sum_{S} |S| \, |f_S|_2^2 \leq b. \tag{1}$$

Then there exists a function g, ± 1-valued, satisfying

$$\mathbf{P}[f \neq g] \leq \epsilon,$$

such that g depends on at most

$$\mathcal{C} = \frac{b^2 2^{12b/\epsilon} + 3}{\epsilon^3 \alpha^{2b/\epsilon + 2}}$$

coordinates. Note that $\mathcal{C}$ does not depend on n.

The proof requires the following straightforward lemma.

Lemma 2 *Let $1 \leq q \leq 2$ and suppose that*

$$\mathbf{P}\left[|f| \in \{0\} \cup [\lambda, +\infty)\right] = 1,$$

where $\lambda > 0$, then

$$|f|_q^q \leq \lambda^{q-2} |f|_2^2.$$

Proof: (Lemma) It is easy to check that the inequality holds pointwise. The result follows. □

We now proceed with the proof of the theorem.

Proof: (Theorem) Let $\gamma > 0$ and

$$J = \{i \,:\, I_i(f) < \gamma\}.$$

Our goal is to show that, for a well chosen value of γ, the set J is such that if

$$h = \sum_{S : S\cap J = \emptyset} f_S$$

then

$$\mathbf{P}[f \neq \mathrm{sgn}(h)] \leq \epsilon.$$

For this, it suffices to show that

$$|h - f|_2^2 \leq \epsilon.$$

Let

$$f_i = \sum_{S : i \in S} f_S.$$

Note first that $|f_i|_2^2 = I_i(f)$. Also, it is easy to see that for all x

$$f_i(x) = f(x) - \mathbf{E}[f \mid x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n],$$

(by expanding both sides over $\{f_S\}_{S\subseteq[n]}$). Therefore, it holds that for all x

$$\text{either } |f_i(x)| \geq 2\alpha \text{ or } f_i(x) = 0. \qquad (2)$$

Let q, η be such that the space $L^2(\mu^n)$ is $(q, 2, \eta)$-hypercontractive (we give actual constants below). Then, by Lemma 2, (2), and the assumptions of the Theorem, we have

$$\begin{aligned}
\sum_S |S \cap J|\, \eta^{2|S|}\, |f_S|_2^2 &= \sum_{i\in J} |T_\eta f_i|_2^2 \\
&\leq \sum_{i \in J} |f_i|_q^2 \\
&\leq \sum_{i\in J} \left((2\alpha)^{q-2} |f_i|_2^2\right)^{2/q} \\
&= (2\alpha)^{2-4/q} \sum_{i \in J} \left(|f_i|_2^2\right)^{2/q} \\
&\leq (2\alpha)^{2-4/q} \gamma^{2/q-1} \sum_i |f_i|_2^2 \\
&\leq (2\alpha)^{2-4/q} \gamma^{2/q-1} b \\
&\equiv t.
\end{aligned}$$

From this we get that

$$\sum \left\{ |f_S|_2^2 \,:\, |S \cap J| \geq \frac{2t\eta^{-2|S|}}{\epsilon} \right\} \leq \frac{\epsilon}{2}. \tag{3}$$

Observe furthermore that it follows from (1) that

$$\sum \left\{ |f_S|_2^2 \,:\, |S| \geq \frac{2b}{\epsilon} \right\} \leq \frac{\epsilon}{2}. \tag{4}$$

Combining (4) and (3), we obtain

$$\sum \left\{ |f_S|_2^2 \,:\, |S \cap J| \geq H \right\} \leq \epsilon,$$

where

$$H = \frac{2t\eta^{-4b/\epsilon}}{\epsilon}.$$

It remains to show that $H < 1$. For this, choose

$$q = \frac{3}{2}, \qquad \eta = \frac{\alpha^{1/6}}{2},$$

and

$$\gamma < \frac{\epsilon^3 \alpha^{2b/\epsilon+2}}{b^3 2^{12b/\epsilon+3}} = \alpha^{O(b/\epsilon)}\epsilon^3.$$

In particular, we have

$$|J^c| \leq \frac{b}{\gamma}.$$

□

Tight Example. The *tribes functions* provide a tight example for Theorem 1. Assume $n = T\,2^T$ for some $T > 0$. A tribe function is a function $f \,:\, \{-1,1\}_0^n \to \{-1,1\}$ defined as follows: think of the n variables as being 2^T "tribes" of T variables; a tribe has value -1 unless all tribe variables are 1; $f = 1$ iff there exists a tribe whose value is 1. It is easy to see that the probability that f is 1 is roughly $1 - 1/e$ (Poisson variable). Also

$$I_i(f) \leq \mathbf{P}[\text{all other tribe variables are 1}] \leq 2^{-T+1} = O\left(\frac{\log n}{n}\right).$$

Therefore the influence sum is $O(\log n)$. The tightness is left as an exercise.

Exercise 3 *(1 pt) Use tribes functions to show that Theorem 1 is tight, i.e. show that for tribes functions, $2^{\Omega(b/\epsilon)}$ coordinates are necessary to get an ϵ-approximation.*

2 Dictators Grow Slowly

In this section, we show that among all monotone functions, the dictator functions are the ones that grow the slowest.

Proposition 4 *Let $f : \{0,1\}^n \to \{0,1\}$ be a monotone function such that*

$$\frac{\mathbf{E}_p[f]}{1-\mathbf{E}_p[f]} = c\frac{p}{1-p},$$

for some $c > 0$. Then if $q > p$,

$$\frac{\mathbf{E}_q[f]}{1-\mathbf{E}_q[f]} \geq c\frac{q}{1-q}.$$

This inequality is satisfied with equality for dictator functions.

Proof: By Russo's formula,

$$\frac{d}{dt}\mathbf{E}^t[f] = \frac{\sum I_i^t(f)}{t(1-t)} \geq \frac{\mathbf{Var}^t[f]}{t(1-t)} = \frac{\mathbf{E}^t(f)(1-\mathbf{E}^t(f))}{t(1-t)}.$$

Solving this differential equation gives immediately

$$\frac{\mathbf{E}_t[f]}{1-\mathbf{E}_t[f]}\frac{1-\mathbf{E}_p[f]}{\mathbf{E}_p[f]} \geq \frac{t}{1-t}\frac{1-p}{p}.$$

□

References

[Fr98] E. Friedgut, Boolean Functions With Low Average Sensitivity Depend On Few Coordinates, Combinatorica, 18(1):27–35, 1998.