Robust Dimension Free Isoperimetry in Gaussian Space

Elchanan Mossel and Joe Neeman (UC Berkeley)

May 10, 2012

The Gaussian Isoperimetric problem

- In Euclidean space: among all sets of volume a, the ball of volume a minimizes the surface area (Steiner 1838, Levy 1919 etc.)

The Gaussian Isoperimetric problem

- In Euclidean space: among all sets of volume *a*, the ball of volume *a* minimizes the surface area (Steiner 1838, Levy 1919 etc.)
- The Gaussian analog of this result is due to B. Tsirelson and V. Sudakov (1974) and independently due to C. Borell (1975).

The Gaussian Isoperimetric problem

- In Euclidean space: among all sets of volume *a*, the ball of volume *a* minimizes the surface area (Steiner 1838, Levy 1919 etc.)
- The Gaussian analog of this result is due to B. Tsirelson and V. Sudakov (1974) and independently due to C. Borell (1975).
- The inequality states that if $A \subset \mathbb{R}^n$ and $B = \{x \in \mathbb{R}^n : x \cdot a \geq b\} \subset \mathbb{R}^n$ is a half-space of the same gaussian measure ($\gamma_n(A) = \gamma_n(B)$) then:

The Gaussian Isoperimetric problem

- In Euclidean space: among all sets of volume a, the ball of volume a minimizes the surface area (Steiner 1838, Levy 1919 etc.)
- The Gaussian analog of this result is due to B. Tsirelson and V. Sudakov (1974) and independently due to C. Borell (1975).
- The inequality states that if $A \subset \mathbb{R}^n$ and $B = \{x \in \mathbb{R}^n : x \cdot a \geq b\} \subset \mathbb{R}^n$ is a half-space of the same gaussian measure ($\gamma_n(A) = \gamma_n(B)$) then:
- $\gamma_n^+(A) \geq \gamma_n^+(B)$ where γ_n^+ is the Gaussian surface area.

The Gaussian Isoperimetric problem

- In Euclidean space: among all sets of volume a, the ball of volume a minimizes the surface area (Steiner 1838, Levy 1919 etc.)
- The Gaussian analog of this result is due to B. Tsirelson and V. Sudakov (1974) and independently due to C. Borell (1975).
- The inequality states that if $A \subset \mathbb{R}^n$ and $B = \{x \in \mathbb{R}^n : x \cdot a \geq b\} \subset \mathbb{R}^n$ is a half-space of the same gaussian measure ($\gamma_n(A) = \gamma_n(B)$) then:
- $\gamma_n^+(A) \geq \gamma_n^+(B)$ where γ_n^+ is the Gaussian surface area.
- $$\gamma_n^+(A) := \liminf_{\epsilon \to 0} \frac{1}{\epsilon}(\gamma_n(A_\epsilon) - \gamma_n(A)), \quad A_\epsilon = \{y : d_2(y, A) \leq \epsilon\}.$$

The Gaussian Isoperimetric problem

- In Euclidean space: among all sets of volume a, the ball of volume a minimizes the surface area (Steiner 1838, Levy 1919 etc.)
- The Gaussian analog of this result is due to B. Tsirelson and V. Sudakov (1974) and independently due to C. Borell (1975).
- The inequality states that if $A \subset \mathbb{R}^n$ and $B = \{x \in \mathbb{R}^n : x \cdot a \geq b\} \subset \mathbb{R}^n$ is a half-space of the same gaussian measure ($\gamma_n(A) = \gamma_n(B)$) then:
- $\gamma_n^+(A) \geq \gamma_n^+(B)$ where γ_n^+ is the Gaussian surface area.
- $$\gamma_n^+(A) := \liminf_{\epsilon \to 0} \frac{1}{\epsilon}(\gamma_n(A_\epsilon) - \gamma_n(A)), \quad A_\epsilon = \{y : d_2(y, A) \leq \epsilon\}.$$
- In other words: $\gamma_n^+(A) \geq I(\gamma_n(A))$, where $I(x) := \varphi(\Phi^{-1}(x))$ and φ, Φ are the Gaussian density, CDF).

Uniqueness and robustness

Natural questions

Uniqueness and robustness

Natural questions

Are half spaces the only minimizers of Gaussian surface area?

Uniqueness and robustness

Natural questions

Are half spaces the only minimizers of Gaussian surface area?

Is A with almost minimal boundary necessarily almost a half space?

Uniqueness and robustness

Natural questions

Are half spaces the only minimizers of Gaussian surface area?

Is A with almost minimal boundary necessarily almost a half space?

The uniqueness and robustness question

Are half spaces the only minimizers of Gaussian surface area?

Is A with almost minimal boundary neccasirly almost a half space?

- Slow progress since the 70s.

The uniqueness and robustness question

Are half spaces the only minimizers of Gaussian surface area?

Is A with almost minimal boundary neccasirly almost a half space?

- Slow progress since the 70s.
- Erhard (86): Uniqueness for nice sets.

The uniqueness and robustness question

Are half spaces the only minimizers of Gaussian surface area?

Is A with almost minimal boundary neccasirly almost a half space?

- Slow progress since the 70s.
- Erhard (86): Uniqueness for nice sets.
- Carlen and Kerce (01): Uniqueness for general sets.

The uniqueness and robustness question

Are half spaces the only minimizers of Gaussian surface area?

Is A with almost minimal boundary neccasirly almost a half space?

- Slow progress since the 70s.
- Erhard (86): Uniqueness for nice sets.
- Carlen and Kerce (01): Uniqueness for general sets.
- Assume $\gamma_n(A) = 0.5$.

The uniqueness and robustness question

Are half spaces the only minimizers of Gaussian surface area?

Is A with almost minimal boundary neccasirly almost a half space?

- Slow progress since the 70s.
- Erhard (86): Uniqueness for nice sets.
- Carlen and Kerce (01): Uniqueness for general sets.
- Assume $\gamma_n(A) = 0.5$.
- Cianchi, Fusco, Maggi, and Pratelli (2011): If $\gamma_n^+(A) \leq I(\gamma_n(A)) + \delta$ then there exists a half space B with $\gamma_n(A \Delta B) \leq c(n)\delta^{1/2}$.

The uniqueness and robustness question

Are half spaces the only minimizers of Gaussian surface area?

Is A with almost minimal boundary neccasirly almost a half space?

- Slow progress since the 70s.
- Erhard (86): Uniqueness for nice sets.
- Carlen and Kerce (01): Uniqueness for general sets.
- Assume $\gamma_n(A) = 0.5$.
- Cianchi, Fusco, Maggi, and Pratelli (2011): If $\gamma_n^+(A) \leq I(\gamma_n(A)) + \delta$ then there exists a half space B with $\gamma_n(A \Delta B) \leq c(n)\delta^{1/2}$.
- No bound on $c(n)$.

The uniqueness and robustness question

Are half spaces the only minimizers of Gaussian surface area?

Is A with almost minimal boundary neccasirly almost a half space?

- Slow progress since the 70s.
- Erhard (86): Uniqueness for nice sets.
- Carlen and Kerce (01): Uniqueness for general sets.
- Assume $\gamma_n(A) = 0.5$.
- Cianchi, Fusco, Maggi, and Pratelli (2011): If $\gamma_n^+(A) \leq I(\gamma_n(A)) + \delta$ then there exists a half space B with $\gamma_n(A \Delta B) \leq c(n)\delta^{1/2}$.
- No bound on $c(n)$.
- M, Neeman (12): If $\gamma_n^+(A) \leq I(A) + \delta$ then there exists a half space with $\gamma_n(A \Delta B) \leq C \log^{-1/6}(1/\delta)$.

Robustness Results

Assume $\gamma_n(A) = 0.5$ and $\gamma_n^+(A) \leq I(A) + \delta$

- Cianchi, Fusco, Maggi, and Pratelli (2011): Exists a half space B with

$$\gamma_n(A \Delta B) \leq c(n)\delta^{1/2}$$

Robustness Results

Assume $\gamma_n(A) = 0.5$ and $\gamma_n^+(A) \leq I(A) + \delta$

- Cianchi, Fusco, Maggi, and Pratelli (2011): Exists a half space B with
$$\gamma_n(A \Delta B) \leq c(n)\delta^{1/2}$$
- No bound on $c(n)$.

Robustness Results

Assume $\gamma_n(A) = 0.5$ and $\gamma_n^+(A) \leq I(A) + \delta$

- Cianchi, Fusco, Maggi, and Pratelli (2011): Exists a half space B with
$$\gamma_n(A\Delta B) \leq c(n)\delta^{1/2}$$
- No bound on $c(n)$.
- M, Neeman (2012): If Exists a half space B with
$$\gamma_n(A\Delta B) \leq C \log^{-1/6}(1/\delta).$$

Robustness Results

Assume $\gamma_n(A) = 0.5$ and $\gamma_n^+(A) \leq I(A) + \delta$

- Cianchi, Fusco, Maggi, and Pratelli (2011): Exists a half space B with
$$\gamma_n(A \Delta B) \leq c(n)\delta^{1/2}$$
- No bound on $c(n)$.
- M, Neeman (2012): If Exists a half space B with
$$\gamma_n(A \Delta B) \leq C \log^{-1/6}(1/\delta).$$
- Natural conjecture: Exists a half space B with
$$\gamma_n(A \Delta B) \leq C\sqrt{\delta}.$$

Techniques

- A common approach is using geometric techniques such as symmetrizations (starting with Steiner 1838!).

Techniques

- A common approach is using geometric techniques such as symmetrizations (starting with Steiner 1838!).
- Our approach follows Bobkov and Ledoux in:

Techniques

- A common approach is using geometric techniques such as symmetrizations (starting with Steiner 1838!).
- Our approach follows Bobkov and Ledoux in:
  - Analyzing a function version of the inequality.

Techniques

- A common approach is using geometric techniques such as symmetrizations (starting with Steiner 1838!).
- Our approach follows Bobkov and Ledoux in:
  - Analyzing a function version of the inequality.
  - Utilizing the semi-group flow.

Bobkov's ineqaulity

Bobkov proved a functional version of the inequality:

- Bobkov: For any smooth function $f : \mathbb{R}^n \to [0,1]$ of bounded variation,
$$I(\mathbb{E}f) \leq \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|_2^2}.$$

Bobkov's ineqaulity

Bobkov proved a functional version of the inequality:

- Bobkov: For any smooth function $f : \mathbb{R}^n \to [0, 1]$ of bounded variation,

$$I(\mathbb{E}f) \leq \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|_2^2}.$$

- Since $I(0) = I(1) = 0$ then one can show that if A is a "nice set" then:

$$I(\gamma_n(A)) \leq \text{"}\mathbb{E}[\|\nabla 1_A\|_2]\text{"} = \gamma_n^+(A)$$

Ledoux' proof of Bobkov's inequality

- Consider the Ornstein-Uhlenbeck semigroup:

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\, d\gamma_n(y).$$

Ledoux' proof of Bobkov's inequality

- Consider the Ornstein-Uhlenbeck semigroup:

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\, d\gamma_n(y).$$

- look at: $\psi(t) := \mathbb{E}\sqrt{I^2(P_t f) + \|\nabla P_t f\|^2}$.

Ledoux' proof of Bobkov's inequality

- Consider the Ornstein-Uhlenbeck semigroup:

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\, d\gamma_n(y).$$

- look at: $\psi(t) := \mathbb{E}\sqrt{I^2(P_t f) + \|\nabla P_t f\|^2}$.
- When $t = 0$: $\psi(0) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2}$

Ledoux' proof of Bobkov's inequality

- Consider the Ornstein-Uhlenbeck semigroup:

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\, d\gamma_n(y).$$

- look at: $\psi(t) := \mathbb{E}\sqrt{I^2(P_t f) + \|\nabla P_t f\|^2}$.
- When $t = 0$: $\psi(0) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2}$
- and when $t = \infty$: $\psi(\infty) = I(\mathbb{E}f)$.

Ledoux' proof of Bobkov's inequality

- Consider the Ornstein-Uhlenbeck semigroup:

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\, d\gamma_n(y).$$

- look at: $\psi(t) := \mathbb{E}\sqrt{I^2(P_t f) + \|\nabla P_t f\|^2}$.
- When $t = 0$: $\psi(0) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2}$
- and when $t = \infty$: $\psi(\infty) = I(\mathbb{E} f)$.
- Suffices to prove ψ_t is decreasing.

Ledoux' proof of Bobkov's inequality

- Consider the Ornstein-Uhlenbeck semigroup:

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}}y)\, d\gamma_n(y).$$

- look at: $\psi(t) := \mathbb{E}\sqrt{I^2(P_t f) + \|\nabla P_t f\|^2}$.
- When $t = 0$: $\psi(0) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2}$
- and when $t = \infty$: $\psi(\infty) = I(\mathbb{E}f)$.
- Suffices to prove ψ_t is decreasing.
- Nice properties that allow to establish $\psi'(t) \leq 0$:
  - $II'' = -1$
  - Integration by parts $\int -fLg\, d\gamma_n = \int \langle \nabla f, \nabla g \rangle d\gamma_n$ (where $Lf(x) = \Delta f(x) - \langle x, \nabla f \rangle$ is the generator).
  - etc.

Carlen and Kerce analysis

- Carlen and Kerce (2001):

Carlen and Kerce analysis

- Carlen and Kerce (2001):
- Let $f : \mathbb{R}^n \to [0,1]$ be smooth. Define $h_t = \Phi^{-1} \circ (P_t f)$ and

$$\delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f).$$

Carlen and Kerce analysis

- Carlen and Kerce (2001):
- Let $f : \mathbb{R}^n \to [0,1]$ be smooth. Define $h_t = \Phi^{-1} \circ (P_t f)$ and

$$\delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f).$$

- Then

$$\delta(f) \geq \int_0^\infty \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}}\, dt,$$

where $H(h_t)$ is the Hessian matrix of h_t and $\|\cdot\|_F$ denotes the Frobenius norm.

Carlen and Kerce analysis

- Carlen and Kerce (2001):
- Let $f : \mathbb{R}^n \to [0,1]$ be smooth. Define $h_t = \Phi^{-1} \circ (P_t f)$ and
$$\delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f).$$
- Then
$$\delta(f) \geq \int_0^\infty \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1 + \|\nabla h_t\|^2)^{3/2}}\,dt,$$
where $H(h_t)$ is the Hessian matrix of h_t and $\|\cdot\|_F$ denotes the Frobenius norm.
- $\delta(f) = 0 \implies h_t$ is linear $t > 0 \implies P_t f$ is Gaussian $\forall t$.

Carlen and Kerce analysis

- Carlen and Kerce (2001):
- Let $f : \mathbb{R}^n \to [0,1]$ be smooth. Define $h_t = \Phi^{-1} \circ (P_t f)$ and

$$\delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f).$$

- Then

$$\delta(f) \geq \int_0^\infty \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1 + \|\nabla h_t\|^2)^{3/2}}\, dt,$$

  where $H(h_t)$ is the Hessian matrix of h_t and $\|\cdot\|_F$ denotes the Frobenius norm.
- $\delta(f) = 0 \implies h_t$ is linear $t > 0 \implies P_t f$ is Gaussian $\forall t$.
- $f = 1_A$ and $\delta(f) = 0$ by limiting arguments f is a half-space.

M + Neeman proof Strategy

- Use the Carlen and Kerce bound.

M + Neeman proof Strategy

- Use the Carlen and Kerce bound.
- Show that if $\delta(f)$ is small then h_t close to linear.

M + Neeman proof Strategy

- Use the Carlen and Kerce bound.
- Show that if $\delta(f)$ is small then h_t close to linear.
- Conclude that f is close to a Gaussian / half-space.

Proof Cartoon

- To demonstrate the main ideas of the proof assume a stronger result than Carlen and Kerce:

$$\delta(f) \geq \int_0^\infty \mathbb{E}\|H(h_t)\|_F^2 \, dt,$$

$$(h_t = \Phi^{-1} \circ (P_t f), \quad \delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f).)$$

Proof Cartoon

- To demonstrate the main ideas of the proof assume a stronger result than Carlen and Kerce:

$$\delta(f) \geq \int_0^\infty \mathbb{E}\|H(h_t)\|_F^2 \, dt,$$

$$(h_t = \Phi^{-1} \circ (P_t f), \quad \delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f).)$$

- If $\delta(f) < \epsilon^2$ then there exists a $t \in [0, \epsilon]$ with $\|H(h_t)\|_F^2 \leq \epsilon$.

Proof Cartoon

- To demonstrate the main ideas of the proof assume a stronger result than Carlen and Kerce:

$$\delta(f) \geq \int_0^\infty \mathbb{E}\|H(h_t)\|_F^2 \, dt,$$

$(h_t = \Phi^{-1} \circ (P_t f), \quad \delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f).)$

- If $\delta(f) < \epsilon^2$ then there exists a $t \in [0, \epsilon]$ with $\|H(h_t)\|_F^2 \leq \epsilon$.
- Second order Poincare inequality: For any twice-differentiable $f \in L_2(\mathbb{R}^n, \gamma_n)$,

$$\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 \leq \mathbb{E}\|H(f)\|_F^2,$$

Proof Cartoon

- To demonstrate the main ideas of the proof assume a stronger result than Carlen and Kerce:

$$\delta(f) \geq \int_0^\infty \mathbb{E}\|H(h_t)\|_F^2 \, dt,$$

$$(h_t = \Phi^{-1} \circ (P_t f), \quad \delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f).)$$

- If $\delta(f) < \epsilon^2$ then there exists a $t \in [0, \epsilon]$ with $\|H(h_t)\|_F^2 \leq \epsilon$.
- Second order Poincare inequality: For any twice-differentiable $f \in L_2(\mathbb{R}^n, \gamma_n)$,

$$\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 \leq \mathbb{E}\|H(f)\|_F^2,$$

- $\implies \mathbb{E}(h_t(x) - a \cdot x - b)^2 \leq \epsilon \implies f_t$ is close to a Gaussian.

Proof Cartoon

- To demonstrate the main ideas of the proof assume a stronger result than Carlen and Kerce:

$$\delta(f) \geq \int_0^\infty \mathbb{E}\|H(h_t)\|_F^2 \, dt,$$

$$(h_t = \Phi^{-1} \circ (P_t f), \quad \delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f).)$$

- If $\delta(f) < \epsilon^2$ then there exists a $t \in [0, \epsilon]$ with $\|H(h_t)\|_F^2 \leq \epsilon$.
- Second order Poincare inequality: For any twice-differentiable $f \in L_2(\mathbb{R}^n, \gamma_n)$,

$$\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 \leq \mathbb{E}\|H(f)\|_F^2,$$

- $\implies \mathbb{E}(h_t(x) - a \cdot x - b)^2 \leq \epsilon \implies f_t$ is close to a Gaussian.
- Now apply P_t^{-1} to obtain that f is close to Gaussian.

Challenges

For the real proof there are a few challenges:

- Need to prove a second order Poincare inequality.

Challenges

For the real proof there are a few challenges:

- Need to prove a second order Poincare inequality.
- P_t^{-1} is not bounded - so cannot simply apply it.

Challenges

For the real proof there are a few challenges:

- Need to prove a second order Poincare inequality.
- P_t^{-1} is not bounded - so cannot simply apply it.
- Main challenge: prove that there exists a t_* such that for $t > t_* - 1$ and f with $\mathbb{E}f \leq 1/2$ it holds that:

$$(*)\ \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \geq cl^2(\mathbb{E}f)\Big(\mathbb{E}\big(\|H(h_t)\|_F^2\big)\Big)^4 \log^{-3}\frac{1}{\mathbb{E}f}.$$

Challenges

For the real proof there are a few challenges:

- Need to prove a second order Poincare inequality.
- P_t^{-1} is not bounded - so cannot simply apply it.
- Main challenge: prove that there exists a t_* such that for $t > t_* - 1$ and f with $\mathbb{E}f \le 1/2$ it holds that:

$$(*)\ \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \ge cI^2(\mathbb{E}f)\Big(\mathbb{E}(\|H(h_t)\|_F^2)\Big)^4 \log^{-3}\frac{1}{\mathbb{E}f}.$$

- Given (*) , if $\delta(f) < \epsilon$ then

$$\int_{t*-1}^{t} \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}}\, dt < \epsilon.$$

  Therefore there exists $t < t*$ such that

$$\mathbb{E}(\|H(h_t)\|_F^2) \le c^{-1}\epsilon^{1/4}\log^{3/4}\frac{1}{\mathbb{E}f} I^{-2}(\mathbb{E}f)$$

Challenges

For the real proof there are a few challenges:

- Need to prove a second order Poincare inequality.
- P_t^{-1} is not bounded - so cannot simply apply it.
- Main challenge: prove that there exists a t_* such that for $t > t_* - 1$ and f with $\mathbb{E}f \leq 1/2$ it holds that:

$$(*)\ \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \geq cI^2(\mathbb{E}f)\Big(\mathbb{E}\big(\|H(h_t)\|_F^2\big)\Big)^4 \log^{-3}\frac{1}{\mathbb{E}f}.$$

- Given (*) , if $\delta(f) < \epsilon$ then

$$\int_{t*-1}^{t} \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}}\, dt < \epsilon.$$

  Therefore there exists $t < t*$ such that

$$\mathbb{E}\big(\|H(h_t)\|_F^2\big) \leq c^{-1}\epsilon^{1/4}\log^{3/4}\frac{1}{\mathbb{E}f}I^{-2}(\mathbb{E}f)$$

- So the main challenge is to prove (*).

A second order Poincare inequality

- 2nd order Poincare inequality: For any twice-differentiable $f \in L_2(\mathbb{R}^n, \gamma_n)$,

$$\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 \leq \mathbb{E}\|H(f)\|_F^2,$$

A second order Poincare inequality

- 2nd order Poincare inequality: For any twice-differentiable $f \in L_2(\mathbb{R}^n, \gamma_n)$,
$$\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 \leq \mathbb{E}\|H(f)\|_F^2,$$
- Almost surely known.

A second order Poincare inequality

- 2nd order Poincare inequality: For any twice-differentiable $f \in L_2(\mathbb{R}^n, \gamma_n)$,

$$\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 \leq \mathbb{E}\|H(f)\|_F^2,$$

- Almost surely known.
- Proof sketch: Hermite Expand: $f = \sum_\alpha b_\alpha H_\alpha$.

A second order Poincare inequality

- 2nd order Poincare inequality: For any twice-differentiable $f \in L_2(\mathbb{R}^n, \gamma_n)$,

$$\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 \leq \mathbb{E}\|H(f)\|_F^2,$$

- Almost surely known.
- Proof sketch: Hermite Expand: $f = \sum_\alpha b_\alpha H_\alpha$.
- $\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 = \sum_{|\alpha| \geq 2} b_\alpha^2 \alpha!$.

A second order Poincare inequality

- 2nd order Poincare inequality: For any twice-differentiable $f \in L_2(\mathbb{R}^n, \gamma_n)$,
$$\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 \leq \mathbb{E}\|H(f)\|_F^2,$$
- Almost surely known.
- Proof sketch: Hermite Expand: $f = \sum_\alpha b_\alpha H_\alpha$.
- $\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 = \sum_{|\alpha| \geq 2} b_\alpha^2 \alpha!$.
-

$$\begin{aligned}
\mathbb{E}\|H(f)\|_F^2 &= \sum_{i,j} \mathbb{E}\Big(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big)^2 \\
&= \sum_{i \neq j} \sum_{\{\alpha : \alpha_i, \alpha_j \geq 1\}} b_\alpha^2 \alpha_i \alpha_j \alpha! + \sum_i \sum_{\{\alpha : \alpha_i \geq 2\}} b_\alpha^2 \alpha_i(\alpha_i - 1)\alpha! \\
&\geq \sum_{|\alpha| \geq 2} b_\alpha^2 \alpha! = \min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2.
\end{aligned}$$

]

Boundedness of P_t^{-1}

- Challenge: Assume $\mathbb{E}(f_t - \Phi(a\dot{x} + b))^2 \leq \epsilon$. Is it true that

$$\min_{a \in [0,\infty]} \mathbb{E}(\|f_t - \Phi(a'\dot{x} + b)\|_2 \leq \epsilon'(t, \epsilon)?$$

Boundedness of P_t^{-1}

- Challenge: Assume $\mathbb{E}(f_t - \Phi(a\dot{x} + b))^2 \leq \epsilon$. Is it true that

$$\min_{a \in [0,\infty]} \mathbb{E}(\|f_t - \Phi(a'\dot{x} + b)\|_2 \leq \epsilon'(t, \epsilon)?$$

- Good news: $P_t^{-1}\Phi(a\dot{x} + b) = \Phi(a'\dot{x} + b)$ when defined.

Boundedness of P_t^{-1}

- Challenge: Assume $\mathbb{E}(f_t - \Phi(a\dot{x} + b))^2 \leq \epsilon$. Is it true that

$$\min_{a \in [0,\infty]} \mathbb{E}(\|f_t - \Phi(a'\dot{x} + b)\|_2 \leq \epsilon'(t, \epsilon)?$$

- Good news: $P_t^{-1}\Phi(a\dot{x} + b) = \Phi(a'\dot{x} + b)$ when defined.
- Good news: $P_t^{-1} H_\alpha = e^{t\|\alpha|} H_\alpha$
- Bad news: P_t^{-1} not bounded!

Boundedness of P_t^{-1}

- Challenge: Assume $\mathbb{E}(f_t - \Phi(a\dot{x} + b))^2 \leq \epsilon$. Is it true that

$$\min_{a \in [0,\infty]} \mathbb{E}(\|f_t - \Phi(a'\dot{x} + b)\|_2 \leq \epsilon'(t,\epsilon)?$$

- Good news: $P_t^{-1}\Phi(a\dot{x} + b) = \Phi(a'\dot{x} + b)$ when defined.
- Good news: $P_t^{-1} H_\alpha = e^{t\|\alpha|} H_\alpha$
- Bad news: P_t^{-1} not bounded!
- The fix: if f is smooth or $f = 1_A$ has small boundary then cannot have too much mass on high coefficients.

Boundedness of P_t^{-1}

- Challenge: Assume $\mathbb{E}(f_t - \Phi(a\dot{x} + b))^2 \leq \epsilon$. Is it true that

$$\min_{a \in [0,\infty]} \mathbb{E}(\|f_t - \Phi(a'\dot{x} + b)\|_2 \leq \epsilon'(t, \epsilon)?$$

- Good news: $P_t^{-1}\Phi(a\dot{x} + b) = \Phi(a'\dot{x} + b)$ when defined.
- Good news: $P_t^{-1} H_\alpha = e^{t\|\alpha|} H_\alpha$
- Bad news: P_t^{-1} not bounded!
- The fix: if f is smooth or $f = 1_A$ has small boundary then cannot have too much mass on high coefficients.
- E.g. by Ledoux (94): $\mathbb{E}f(f - P_t f) \leq c\sqrt{t}\mathbb{E}\|\nabla f\|$.

Boundedness of P_t^{-1}

- Challenge: Assume $\mathbb{E}(f_t - \Phi(a\dot{x} + b))^2 \leq \epsilon$. Is it true that

$$\min_{a \in [0,\infty]} \mathbb{E}(\|f_t - \Phi(a'\dot{x} + b)\|_2 \leq \epsilon'(t, \epsilon)?$$

- Good news: $P_t^{-1}\Phi(a\dot{x} + b) = \Phi(a'\dot{x} + b)$ when defined.
- Good news: $P_t^{-1} H_\alpha = e^{t\|\alpha|} H_\alpha$
- Bad news: P_t^{-1} not bounded!
- The fix: if f is smooth or $f = 1_A$ has small boundary then cannot have too much mass on high coefficients.
- E.g. by Ledoux (94): $\mathbb{E}f(f - P_t f) \leq c\sqrt{t}\mathbb{E}\|\nabla f\|$.
- If $\mathbb{E}\|\nabla f\| \geq 10$ then

$$\delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f) \geq \mathbb{E}\|\nabla f\| - I(\mathbb{E}f) \geq 9.$$

Boundedness of P_t^{-1}

- Challenge: Assume $\mathbb{E}(f_t - \Phi(a\dot{x} + b))^2 \leq \epsilon$. Is it true that

$$\min_{a \in [0,\infty]} \mathbb{E}(\|f_t - \Phi(a'\dot{x} + b)\|_2 \leq \epsilon'(t,\epsilon)?$$

- Good news: $P_t^{-1}\Phi(a\dot{x} + b) = \Phi(a'\dot{x} + b)$ when defined.
- Good news: $P_t^{-1}H_\alpha = e^{t\|\alpha|}H_\alpha$
- Bad news: P_t^{-1} not bounded!
- The fix: if f is smooth or $f = 1_A$ has small boundary then cannot have too much mass on high coefficients.
- E.g. by Ledoux (94): $\mathbb{E}f(f - P_t f) \leq c\sqrt{t}\mathbb{E}\|\nabla f\|$.
- If $\mathbb{E}\|\nabla f\| \geq 10$ then

$$\delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f) \geq \mathbb{E}\|\nabla f\| - I(\mathbb{E}f) \geq 9.$$

- Similar arguments for sets.

The main challenge

- Need to prove that for f taking values in $[0,1]$ and $\mathbb{E}f \leq 1/2$:

$$(*)\ \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \geq c(\mathbb{E}f)\Big(\mathbb{E}(\|H(h_t)\|_F^2)\Big)^4$$

The main challenge

- Need to prove that for f taking values in $[0,1]$ and $\mathbb{E}f \leq 1/2$:

$$(*)\ \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \geq c(\mathbb{E}f)\Big(\mathbb{E}(\|H(h_t)\|_F^2)\Big)^4$$

- For this, using the reverse log Sobolev inequality we prove that for t large enough:

$$(**)\ \|\nabla h_t\| \leq 1 \text{ a.s. }, \quad \|\nabla f_t\| \leq \frac{\sqrt{2}e^{-t}}{\sqrt{1-e^{-2t}}} f_t \sqrt{\log\frac{1}{f_t}} \text{ a.s}$$

The main challenge

- Need to prove that for f taking values in $[0,1]$ and $\mathbb{E}f \le 1/2$:

$$(*)\ \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \ge c(\mathbb{E}f)\Big(\mathbb{E}(\|H(h_t)\|_F^2)\Big)^4$$

- For this, using the reverse log Sobolev inequality we prove that for t large enough:

$$(**)\ \|\nabla h_t\| \le 1 \text{ a.s. }, \quad \|\nabla f_t\| \le \frac{\sqrt{2}e^{-t}}{\sqrt{1-e^{-2t}}} f_t\sqrt{\log\frac{1}{f_t}} \text{ a.s}$$

- We then use (**), the concavity of I, the reverse-Hölder inequality, and reverse hyper-contractivity to show that

$$(***)\ \mathbb{E}\big(\varphi(h_t)\|H(h_t)\|_F^2\big) \ge cI^2(\mathbb{E}f)\,(\mathbb{E}\|H(h_t)\|_F)^2$$

The main challenge

- Need to prove that for f taking values in $[0,1]$ and $\mathbb{E}f \leq 1/2$:

$$(*)\ \mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \geq c(\mathbb{E}f)\Big(\mathbb{E}(\|H(h_t)\|_F^2)\Big)^4$$

- For this, using the reverse log Sobolev inequality we prove that for t large enough:

$$(**)\ \|\nabla h_t\| \leq 1 \text{ a.s. }, \quad \|\nabla f_t\| \leq \frac{\sqrt{2}e^{-t}}{\sqrt{1-e^{-2t}}} f_t \sqrt{\log\frac{1}{f_t}} \text{ a.s}$$

- We then use (**), the concavity of I, the reverse-Hölder inequality, and reverse hyper-contractivity to show that

$$(***)\ \mathbb{E}\big(\varphi(h_t)\|H(h_t)\|_F^2\big) \geq cI^2(\mathbb{E}f)\,(\mathbb{E}\|H(h_t)\|_F)^2$$

- Finally using almost all of the tools before and additionally concentration of measure and Hanson-Wright inequalities we prove that for t large enough

$$(****)\ (\mathbb{E}\|H(h_t)\|_F^3)^{1/3} \leq \sqrt{\log(1/(\mathbb{E}f))}$$

Combining the pieces

- By (**) for some t_* and $t > t_*$:

$$\mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \geq c\mathbb{E}\big(\varphi(h_t)\|H(h_t)\|_F^2\big)$$

Combining the pieces

- By (**) for some t_* and $t > t_*$:

$$\mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \geq c\mathbb{E}\big(\varphi(h_t)\|H(h_t)\|_F^2\big)$$

- By (***) for $t > t_*$:

$$\mathbb{E}\big(\varphi(h_t)\|H(h_t)\|_F^2\big) \geq c(\mathbb{E}f)\,(\mathbb{E}\|H(h_t)\|_F)^2\,.$$

Combining the pieces

- By (**) for some t_* and $t > t_*$:

$$\mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \geq c\mathbb{E}\big(\varphi(h_t)\|H(h_t)\|_F^2\big)$$

- By (***) for $t > t_*$:

$$\mathbb{E}\big(\varphi(h_t)\|H(h_t)\|_F^2\big) \geq c(\mathbb{E}f)\left(\mathbb{E}\|H(h_t)\|_F\right)^2.$$

- Sadly - the square is outside the expectation.

Combining the pieces

- By (**) for some t_* and $t > t_*$:

$$\mathbb{E}\frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \geq c\mathbb{E}\big(\varphi(h_t)\|H(h_t)\|_F^2\big)$$

- By (***) for $t > t_*$:

$$\mathbb{E}\big(\varphi(h_t)\|H(h_t)\|_F^2\big) \geq c(\mathbb{E}f)\,(\mathbb{E}\|H(h_t)\|_F)^2\,.$$

- Sadly - the square is outside the expectation.
- However by Hölder's inequality

$$\mathbb{E}\big(\|H(h_t)\|_F^2\big) \leq \Big(\mathbb{E}\|H(h_t)\|_F\Big)^{1/2}\big(\mathbb{E}\|H(h_t)\|_F^3\big)^{1/2}$$

and therefore by (****) the upper bound on $\mathbb{E}\|H(h_t)\|_F$ yields an upper bound on $\mathbb{E}\left(\|H(h_t)\|_F^2\right)$.

Open Problems

- Prove that if $f = 1_A$ satisfies $\delta(f) \leq \delta$ then there exists a half space B such that $\gamma_n(A \Delta B) \leq C\delta^{1/2}$.

Open Problems

- Prove that if $f = 1_A$ satisfies $\delta(f) \leq \delta$ then there exists a half space B such that $\gamma_n(A \Delta B) \leq C\delta^{1/2}$.
- Analyze equality case and robustness of isoperimetric problems for other log-concave measures.

Current and Future Work (M+Neeman)

- Borell showed that if $\gamma_n(B) = \gamma_n(A)$ and B is a half-space then

$$(+)\ \mathbb{E}[1_A P_t 1_A] \leq \mathbb{E}[1_B P_t 1_B]$$

Current and Future Work (M+Neeman)

- Borell showed that if $\gamma_n(B) = \gamma_n(A)$ and B is a half-space then

$$(+)\ \mathbb{E}[1_A P_t 1_A] \leq \mathbb{E}[1_B P_t 1_B]$$

- Say something about the proofs.

Current and Future Work (M+Neeman)

- Borell showed that if $\gamma_n(B) = \gamma_n(A)$ and B is a half-space then
$$(+)\ \mathbb{E}[1_A P_t 1_A] \leq \mathbb{E}[1_B P_t 1_B]$$
- Say something about the proofs.
- Problem: Are half-spaces the only optimizers?

Current and Future Work (M+Neeman)

- Borell showed that if $\gamma_n(B) = \gamma_n(A)$ and B is a half-space then
$$(+)\ \mathbb{E}[1_A P_t 1_A] \leq \mathbb{E}[1_B P_t 1_B]$$
- Say something about the proofs.
- Problem: Are half-spaces the only optimizers?
- Problem: Is there a robust version?

Current and Future Work (M+Neeman)

- Borell showed that if $\gamma_n(B) = \gamma_n(A)$ and B is a half-space then
$$(+)\ \mathbb{E}[1_A P_t 1_A] \leq \mathbb{E}[1_B P_t 1_B]$$
- Say something about the proofs.
- Problem: Are half-spaces the only optimizers?
- Problem: Is there a robust version?
- A's: Yes, Yes (M + Neeman, 2012-3).