Reverse Hyper Contractive Inequalities:
What?
Why??
When???
How????

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Hyper-Contractive Inequalities

The "noise" (Bonami-Beckner) semi-group on $\{-1,1\}^n_{\frac{1}{2}}$

Def 1: T_t is the Markov operator where the conditional probability of y given x is given by $y_i = x_i$ with probability $\frac{1}{2}(1+e^{-t})$ independently for each coordinate

Def 2: T_t is the n'th tensor of the operator T'_t on $L^2(\{-1,1\}_{\frac{1}{2}})$ defined by $T'_t(f) = e^{-t}f + (1-e^{-t})\mathbb{E}[f]$

Hyper-contractivity (Bomani, Beckner, Gross):

For $f : \{-1,1\}^n_{\frac{1}{2}} \to \mathbb{R}$:

$$\|T_t f\|_p \leq \|f\|_q \text{ for } p > q > 1 \text{ and } t \geq \frac{1}{2}\ln\frac{p-1}{q-1}$$

Why do we like Hypercontractive inequalities?

Hyper-contractivity (Bomani, Beckner, Gross):

For $f : \{-1,1\}^n_{\frac{1}{2}} \to \mathbb{R}$:

$$\|T_t f\|_p \leq \|f\|_q \text{ for } p > q > 1 \text{ and } t \geq \frac{1}{2} \ln \frac{p-1}{q-1}$$

Boolean applications

Def: $f : \{-1,1\}^n \to \mathbb{R}$ is *Boolean* if $f \in \{0,1\}$

For Boolean f: relate Fourier decay $\|T_t f\|_2^2 = \sum_S \hat{f}^2(S) e^{-2t|S|}$ to probability of events: $\|f\|_q = \mathbb{P}[f = 1]^{1/q}$

Used in KKL88, Talagrand 94 and most papers in discrete Fourier analysis

Other Discrete Spaces

Other Discrete Spaces

What if $f : \{0,1\}^n_\alpha \to \{0,1\}$?

What if $f : \Omega^n \to \{0,1\}$ for some discrete space Ω?

Many results in discrete Fourier analysis generalize easily given hyper-contractive estimates

Bonami Beckner Noise for General Product Spaces

Setup: $T'f = e^{-t}f + (1-e^{-t})\mathbb{E}[f]$ and look at $T'^{\otimes n}$

Probabilistic meaning: Markov Chain where each coordinate is de-randomized with probability $(1-e^{-t})$ independently

Hyper-contractive estimates for discrete spaces

Oleszkiewicz 2003:

Formula for $\Omega = \{0,1\}_\alpha$ if $p = 2 > q$ or $p < 2 = q$

In particular as $\alpha \to 0$:

$$\|T_t f\|_2 \leq \|f\|_q \text{ iff } t \gtrsim (\tfrac{1}{q} - \tfrac{1}{2}) \ln(1/\alpha)$$

$$\|T_t f\|_p \leq \|f\|_2 \text{ iff } t \gtrsim (\tfrac{1}{2} - \tfrac{1}{p}) \ln(1/\alpha)$$

Wolff 2007:

Let $\alpha = \min_{\omega \in \Omega} \mathbb{P}(\omega)$

As $\alpha \to 0$:

$$\|T_t f\|_p \leq \|f\|_q \text{ iff } t \gtrsim (\tfrac{1}{q} - \tfrac{1}{p}) \ln(1/\alpha)$$

Reverse-Hyper-Contractive inequalities; What?

Borell's reverse bound

Reverse-Hyper-Contractivity (Borell 85):

For all $f : \{-1,1\}^n_{1/2} \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \frac{1}{2} \ln \frac{1-q}{1-p}$$

Reverse-Hyper-Contractive inequalities; What?

Reverse-Hyper-Contractivity (Borell 85)

For all $f : \{-1,1\}^n_{1/2} \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \frac{1}{2} \ln \frac{1-q}{1-p}$$

Does this make sense?

"Norms" when $p, q < 1$

Reverse Minkowski inequality

If f is non-negative $p < 1$ and T is a Markov operator then:

$$\|Tf\|_p \geq \|f\|_p$$

Reverse Hölder inequality

If f, g are non-negative and p, p' are dual norms < 1 then:

$$\mathbb{E}[fg] = \|fg\|_1 \geq \|f\|_p \|g\|_{p'}.$$

Reverse-Hyper-contractivity (Borell 85)

For all $f : \{-1, 1\}^n_{1/2} \rightarrow \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \frac{1}{2} \ln \frac{1-q}{1-p}$$

Why? What? When?

Reverse-Hyper-contractivity (Borell 85)

For all $f : \{-1,1\}^n_{1/2} \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \frac{1}{2}\ln\frac{1-q}{1-p}$$

Why is it true?

What is it good for?

Is it true for other discrete spaces?

Why is it true? Borell's argument

I. Tensor

From Reverse Minkowski suffices to prove for $n = 1$

II. Duality

From Revese Hölder suffices to prove for $0 < q < p < 1$

III. Core of Proof

Write $f(x) = 1 + ax$ where $-1 < a < 1$

Taylor expand $\|f\|_p^p$ and $\|T_t f\|_q^p$ and compare terms

Comment

Steps I. and II. are standard and general. Step III. is at the core of the proof

Borell's argument continued

$n = 1, \quad f(x) = 1 + ax, \quad a \in (-1, 1), \quad 0 < q < p < 1$

$$\|f\|_p^p = 1 + \sum_{n=1}^{\infty} \binom{p}{2n} a^{2n}$$

$$\|T_t f\|_q^q = 1 + \sum_{n=1}^{\infty} \binom{q}{2n} e^{-2nt} a^{2n}$$

By Convexity: $\|T_t f\|_q^p \geq 1 + \frac{p}{q} \sum_{n=1}^{\infty} \binom{q}{2n} e^{-2nt} a^{2n}$

Reduces to: $\frac{p}{q} \binom{q}{2n} e^{-2nt} \geq \binom{p}{2n}$

Or: $\frac{p}{q} \binom{q}{2n} \left(\frac{p-1}{q-1}\right)^n \geq \binom{p}{2n}$

Further reduces to: $(i - q)(1 - p)^{1/2} \leq (i - p)(1 - q)^{1/2}$ for $i \geq 2$

What is it good for?

Correlated pairs (M-O'Donnell-Regev-Steif-Sudakov-05):

Let $x, y \in \{-1, 1\}^n_{1/2}$ correlated as follows:

x is chosen uniformly and y is T_t correlated version.

i.e. $\mathbb{E}[x_i y_i] = e^{-t}$ for all i independently

Let $A, B \subset \{-1, 1\}^n_{1/2}$ with $\mathbb{P}[A] \geq \epsilon$ and $\mathbb{P}[B] \geq \epsilon$

Then: $\mathbb{P}[x \in A, y \in B] \geq \epsilon^{\frac{2}{1-e^{-t}}}$

Pf Sketch:

From Reverse Hölder and Borell's result get for any $f, g > 0$:

$$\mathbb{E}[g T_t f] \geq \|f\|_p \|g\|_q \qquad \forall t \geq \frac{1}{2} \ln \frac{1-q}{1-p}$$

For $f = 1_A, g = 1_B$ optimize norms p and q

What is it good for? Cosmic coin model

Coin-Tossing Model

k players want to toss the same coin

Each player gets a y^i that is ρ correlated with $x \in \{-1,1\}^n$.

If $f(y) \in \{0,1\}$ is the coin toss then prob. of agreement is $\|T_t f\|_k^k + \|T_t(1-f)\|_k^k$.

Proposition (M-O'Donnell-Regev-Steif-Sudakov-05):

If $f \in \{0,1\}$ and $\mathbb{E}[f] \leq 1/2$ then $\|T_t f\|_k^k \leq k^{1-e^{2t}+o(1)}$.

Comments

This is tight!

Proof uses reverse-hyper-contraction. Standard Hyper-contractivity gives a bound of $\|T_t f\|_k^k \leq 0.5^{\frac{1}{\rho^2}}$

Arrow Theorem

Setup

3 alternatives a, b, c that are ranked by n voters.

Voter i preference of a vs. b, b vs. c and c vs. a are x_i, y_i and z_i

$(x_i, y_i, z_i) \in R := \{-1, 1\}^3 \setminus \{(1, 1, 1), (-1, -1, -1)\}$ for all i.

$f, g, h : \{-1, 1\}^n \to \{-1, 1\}$ aggregate the pairwise preference.

Arrow's Theorem (51)

Let $f, g, h : \{-1, 1\}^n \to \{-1, 1\}$ satisfying for $b = \pm 1$:
$f(b, \ldots, b) = g(b, \ldots, b) = h(b, \ldots, b) = b$ (Unanimity).

Then either $\exists i$ s.t $\forall x$, $f(x) = g(x) = h(x) = x_i$ or

$\exists x, y, z \in R^n$ s.t. $f(x) = g(y) = h(z)$.

Arrow Theorem and Hyper-Contractivity

Reverse-hyper-contractivity is essential in recent quantitative proofs of Arrow's theorem by M-2011 and Keller-2011 following Kalai's paper (02) in the balanced case. For example:

Barbera's Lemma (82);

Lef $f, g, h : \{-1,1\}^n \to \{-1,1\}$ with $I_1(f) > 0, I_2(g) > 0$

Then $\exists x, y, z \in R^n$ s.t. $f(x) = g(y) = h(z)$

Quantitative Barbera's Lemma (M-11)

Lef $f, g, h : \{-1,1\}^n \to \{-1,1\}$ with $I_1(f) > \epsilon, I_2(g) > \epsilon$

Then $\mathbb{P}[f(x) = g(y) = h(z) | (x, y, z) \in R^n\}] \geq \frac{\epsilon^3}{36}$

Interest in Rev. Hyp contraction on other spaces?

Motivation 1:

Lower bounds for $\mathbb{P}[X \in A, Y \in B]$ for correlated X, Y in general product spcaes

Motivation 2:

Obtain bounds for "Cosmic Die Problem"

Motivation 3

Prove Quantitative Arrow Theorem for non-uniform distributions over voters profile

Hyper and Reverse-Hyper Contractive inequalities

Reverse-Hyper-contractivity (Borell 85)

For all $f : \{-1,1\}^n_{\frac{1}{2}} \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \frac{1}{2} \ln \frac{1-q}{1-p}$$

Hyper-contractivity (Bomani, Beckner, Gross):

For $f : \{-1,1\}^n_{\frac{1}{2}} \to \mathbb{R}$:

$$\|T_t f\|_p \leq \|f\|_q \text{ for } p > q > 1 \text{ and } t \geq \frac{1}{2} \ln \frac{p-1}{q-1}$$

Question

Is Hyper-Contraction equivalent to Rev.-Hyper-Contraction?

Our Results (M-Oleszkiewicz-Sen-11)

Our Result:

Let Ω be an arbitrary space and let T_t be the corresponding Bonami-Backner semi-group. Then for all $f : \Omega^n \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \ln \frac{1-q}{1-p}$$

Our Results (M-Oleszkiewicz-Sen-11)

Our Result:

For all $f : \Omega^n \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \ln \frac{1-q}{1-p}$$

Compare to Borell 82, Oleszkiewicz 03, Wolff 07

For all $f : \{-1,1\}^n_{1/2} \to \mathbb{R}_+$:

$$\|T_t f\|_q \geq \|f\|_p \text{ for } 1 > p > q \text{ and } t \geq \frac{1}{2} \ln \frac{1-q}{1-p}$$

For all $f : \{-1,1\}^n_\alpha \to \mathbb{R}_+$ need $t \gtrsim (\frac{1}{q} - \frac{1}{p}) \ln \alpha$.

Comments:

Note: inequality does not depend on underlying space!.

Sharper (but not tight) bounds are obtained in the paper

Further Results (M-Oleszkiewicz-Sen-11)

Log-Sobolev and Rev. Hyper-Contraction

Let $T_t = e^{-tL}$ be a general Markov semi-group. Suppose L satisfies 2-Logsob or 1-Logsob inequality with constant C. Then for all $q < p < 1$, all positive f and all $t \geq \frac{C}{4} \log \frac{1-q}{1-p}$ it holds that $\|T_t f\|_q \geq \|f\|_p$.

Product Space Applications (M-Oleszkiewicz-Sen-11)

Correlated Pairs

Let $x, y \in (\Omega, \mu^n)$ correlated as follows:

$x \sim \mu^n$ and y is T_t correlated version where $T_t = e^{-t(I-\mathbb{E})}$ is the Bonami-Beckner operator.

Let $A, B \subset \Omega^n$ with $\mathbb{P}[A] \geq \epsilon$ and $\mathbb{P}[B] \geq \epsilon$. Then:

$$\mathbb{P}[x \in A, y \in B] \geq \epsilon^{\frac{2}{1-e^{-t/2}}}$$

Quantitative Arrow's Theorem

Obtain a quantitative Arrow theorem for voting distribution μ^n where μ is any non-degenerate distribution on $\{-1, 1\}^3 \setminus \{(1, 1, 1), (-1, -1, -1)\}$.

Markov Chain Applications (M-Oleszkiewicz-Sen-11)

Correlated Pairs

Let $x, y \in (\Omega, \mu)$ correlated as follows:

$x \sim \mu$ and y is T_t correlated version where $T_t = e^{-tL}$, where L satisfies 1 or 2-LogSob inequality with constant C

Let $A, B \subset \Omega^n$ with $\mathbb{P}[A] \geq \epsilon$ and $\mathbb{P}[B] \geq \epsilon$. Then:

$$\mathbb{P}[x \in A, y \in B] \geq \epsilon^{\frac{2}{1-e^{-2t/C}}}$$

Glauber Dynamics for Ising model in High Temperatures in $[n]^d$.

$A = \{x : Maj(x) = +\}, \quad , B = \{x : Maj(x) = -\}, \quad C = \Theta(1), \quad t_{mix} = O(\log n)$

Random-Transposition Card Shuffle

General A, B. We have $C = \Theta(n), t_{mix} = \Theta(n \log n)$

A Queueing Theory Application (M-Oleszkiewicz-Sen-11)

A queueing process

Take $\{0,1\}^n_{\frac{\lambda}{n}}$ with the Bonami-Beckner operator T_t and $X = \sum_{i=1}^n X_i$.

As $n \to \infty, X \sim Poisson(\lambda)$ and $T_t X$ is the following queueing process:

At $[t, t+dt]$: 1) Each customer is serviced with probability dt.

2) The probability of a new customer arriving is λdt

From our results if $\mathbb{P}[A] > \epsilon, \mathbb{P}[B] > \epsilon$ then $\mathbb{P}[X \in A, T_t X \in B] \geq \epsilon^{\frac{2}{1-e^{-t/2}}}$

Process has infinite mixing time and 2-logSob. 1-logSob is known to be finite (Liming Wu 97)

Proof Sketch

Equivalence with Log-Sobolev inequalities

Using Equivalence of Log-Sob-p inequality and Hyper/Rev-Hyper inequalities work with Log-Sob-p

Main step: Monotonicity of Log-Sob

Log-Sob-p $\implies$ Log-Sob-q for all $2 \geq p > q \geq 0$

Log-Sob-1 for simple operators

Show that Log-Sob-1 holds with $C = 4$ for the semi-group $e^{-t(I-\mathbb{E})}$

Also: Wu, Bobkov-Ledoux, Diaconis-Saloff-Coste

Definition of logSob

Standard Definitions

$$Ent(f) = \mathbb{E}(f \log f) - \mathbb{E}f \cdot \log \mathbb{E}f$$

$$\mathcal{E}(f,g) = \mathbb{E}(fLg) = \mathbb{E}(gLf) = \mathcal{E}(g,f) = -\frac{d}{dt}\mathbb{E}fT_t g\Big|_{t=0}.$$

Definition of Log-Sob

$$p\text{-logSob(C)} \iff \forall f, Ent(f^p) \le \frac{Cp^2}{4(p-1)}\mathcal{E}(f^{p-1}, f) \ (p \ne 0, 1)$$

$$1\text{-logSob(C)} \iff \forall f, Ent(f) \le \frac{C}{4}\mathcal{E}(f, \log f)$$

$$0\text{-logSob(C)} \iff \forall f, Var(\log f) \le -\frac{C}{2}\mathcal{E}(f, 1/f)$$

Notes

All functions are positive. Non-Standard normalization, 1-logSob $\sim$ modified-logSob (Defined by Bakry, Wu mid 90s)

LogSob - Easy facts

Self-Dual Definition

$$\text{For } p \neq 0, 1\text{: } \mathcal{E}(f^{p-1}, f) = \mathcal{E}(g^{1/p}, g^{p'}), \quad g = f^p$$

$$p\text{-logSob(C)} \iff \forall g, Ent(g) \leq \frac{Cpp'}{4}\mathcal{E}(g^{1/p}, g^{1/p'})$$

$$\implies (p\text{-logSob(C)} \iff p'\text{-logSob(C)}).$$

1-logSob

Claim: If $L = I - \mathbb{E}$ then L is 1-logSob(4).

$$Ent(f) = \mathbb{E}f \log f - \mathbb{E}f \cdot \log \mathbb{E}f \leq \mathbb{E}f \log f - \mathbb{E}f \cdot \mathbb{E} \log f = \mathbb{E}f(\log f - \mathbb{E} \log f) = \mathbb{E}fL \log f = \mathcal{E}(f, \log f).$$

Log-Sob Monotonicity

Main Thm: Monotonicity

p-logSob(C) $\implies$ q-logSob(C) for $0 \leq q \leq p \leq 2$

Log-Sob Monotonicity

Thm: Monotonicity

$$p\text{-logSob(C)} \implies q\text{-logSob(C)} \text{ for } 0 \le q \le p \le 2$$

Main applications

$p = 1, q < 1$ Gives $I - \mathbb{E}$ satisfies Rev. Hyp. Contraction

$p = 2, q < 1$ Gives Hyp. Contraction $\implies$ Rev. Hyp. Contraction

Comments

$1 = q \le p = 2$ is due to Gross ($q > 1$), Bakry ($q = 1$) etc.

Recall p-logSob(C) $\iff \forall g, Ent(g) \le \frac{Cpp'}{4}\mathcal{E}(g^{1/p}, g^{1/p'})$

Using continuity at 0 and 1 suffices to show the following

Comparison of Dirichlet forms

Thm: Generalized Stroock-Varopoulos

For all $p > q$ with $p, q \in (0, 2] \setminus \{1\}$ and all $g > 0$:

$$qq'\mathcal{E}(g^{1/q}, g^{1/q'}) \geq pp'\mathcal{E}(g^{1/p}, g^{1/p'})$$

Comparison of Dirichlet forms

Thm: Generalized Stroock-Varopoulos

For all $p > q$ with $p, q \in (0, 2] \setminus \{1\}$ and all $g > 0$:

$$qq'\mathcal{E}(g^{1/q}, g^{1/q'}) \geq pp'\mathcal{E}(g^{1/p}, g^{1/p'})$$

Proof based on the following Lemma (Exercise):

$\forall a, b > 0$:

$$qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq pp'(a^{1/p} - b^{1/p})(a^{1/p'} - b^{1/p'})$$

Pf Sketch

$$qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq$$
$$pp'(a^{1/p} - b^{1/p})(a^{1/p'} - b^{1/p'}) \implies_{a->g+\text{apply}T_t}$$
$$qq'T_t[(g^{1/q} - b^{1/q})(g^{1/q'} - b^{1/q'})] \geq$$
$$pp'T_t[(g^{1/p} - b^{1/p})(g^{1/p'} - b^{1/p'})] \implies \text{Rearrange}$$

Pf Sketch

$$qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq$$
$$pp'(a^{1/p} - b^{1/p})(a^{1/p'} - b^{1/p'}) \implies_{a->g+\text{apply}\, T_t}$$
$$qq' T_t[(g^{1/q} - b^{1/q})(g^{1/q'} - b^{1/q'})] \geq$$
$$pp' T_t[(g^{1/p} - b^{1/p})(g^{1/p'} - b^{1/p'})] \implies \text{Rearrange}$$
$$qq'\left(T_t[g] - b^{1/q} T_t[g^{1/q'}] - b^{1/q'} T_t[g^{1/q}] + b\right) \geq$$
$$pp'\left(T_t[g] - b^{1/p} T_t[g^{1/p'}] - b^{1/p'} T_t[g^{1/p}] + b\right) \implies_{b->g}$$

Pf Sketch

$$qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq$$
$$pp'(a^{1/p} - b^{1/p})(a^{1/p'} - b^{1/p'}) \implies_{a->g+\text{apply}\, T_t}$$
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$$pp'T_t[(g^{1/p} - b^{1/p})(g^{1/p'} - b^{1/p'})] \implies \text{Rearrange}$$
$$qq'\left(T_t[g] - b^{1/q}T_t[g^{1/q'}] - b^{1/q'}T_t[g^{1/q}] + b\right) \geq$$
$$pp'\left(T_t[g] - b^{1/p}T_t[g^{1/p'}] - b^{1/p'}T_t[g^{1/p}] + b\right) \implies_{b->g}$$
$$qq'\left(T_t[g] - g^{1/q}T_t[g^{1/q'}] - g^{1/q'}T_t[g^{1/q}] + g\right) \geq$$
$$pp'\left(T_t[g] - g^{1/p}T_t[g^{1/p'}] - g^{1/p'}T_t[g^{1/p}] + g\right) \implies \text{Taking } \mathbb{E}$$

Pf Sketch

$$qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq$$
$$pp'(a^{1/p} - b^{1/p})(a^{1/p'} - b^{1/p'}) \implies_{a->g+\text{apply}T_t}$$
$$qq' T_t[(g^{1/q} - b^{1/q})(g^{1/q'} - b^{1/q'})] \geq$$
$$pp' T_t[(g^{1/p} - b^{1/p})(g^{1/p'} - b^{1/p'})] \implies \text{Rearrange}$$
$$qq'\left(T_t[g] - b^{1/q} T_t[g^{1/q'}] - b^{1/q'} T_t[g^{1/q}] + b\right) \geq$$
$$pp'\left(T_t[g] - b^{1/p} T_t[g^{1/p'}] - b^{1/p'} T_t[g^{1/p}] + b\right) \implies_{b->g}$$
$$qq'\left(T_t[g] - g^{1/q} T_t[g^{1/q'}] - g^{1/q'} T_t[g^{1/q}] + g\right) \geq$$
$$pp'\left(T_t[g] - g^{1/p} T_t[g^{1/p'}] - g^{1/p'} T_t[g^{1/p}] + g\right) \implies \text{Taking } \mathbb{E}$$
$$qq'\left(2\mathbb{E}[g] - 2\mathbb{E}[g^{1/q} T_t[g^{1/q'}]]\right) \geq$$
$$pp'\left(2\mathbb{E}[g] - 2\mathbb{E}[g^{1/p} T_t[g^{1/p'}]]\right) \implies$$

Pf Sketch

$qq'(a^{1/q} - b^{1/q})(a^{1/q'} - b^{1/q'}) \geq$
$pp'(a^{1/p} - b^{1/p})(a^{1/p'} - b^{1/p'}) \implies_{a->g+\text{apply}\, T_t}$
$qq' T_t[(g^{1/q} - b^{1/q})(g^{1/q'} - b^{1/q'})] \geq$
$pp' T_t[(g^{1/p} - b^{1/p})(g^{1/p'} - b^{1/p'})] \implies$ Rearrange
$qq'\left(T_t[g] - b^{1/q} T_t[g^{1/q'}] - b^{1/q'} T_t[g^{1/q}] + b\right) \geq$
$pp'\left(T_t[g] - b^{1/p} T_t[g^{1/p'}] - b^{1/p'} T_t[g^{1/p}] + b\right) \implies_{b->g}$
$qq'\left(T_t[g] - g^{1/q} T_t[g^{1/q'}] - g^{1/q'} T_t[g^{1/q}] + g\right) \geq$
$pp'\left(T_t[g] - g^{1/p} T_t[g^{1/p'}] - g^{1/p'} T_t[g^{1/p}] + g\right) \implies$ Taking $\mathbb{E}$
$qq'\left(2\mathbb{E}[g] - 2\mathbb{E}[g^{1/q} T_t[g^{1/q'}]]\right) \geq$
$pp'\left(2\mathbb{E}[g] - 2\mathbb{E}[g^{1/p} T_t[g^{1/p'}]]\right) \implies$
Note that $t = 0$ equality holds. Therefore
$\frac{d}{dt}_{|t=0} LHS \geq \frac{d}{dt}_{|t=0} RHS \implies$
$qq'\mathcal{E}(g^{1/q}, g^{1/q'}) \geq pp'\mathcal{E}(g^{1/p}, g^{1/p'})$

Log-Sob $\iff$ (Rev) Hyper-contraction

Log Sob $\implies$ Rev-Hyper-Contraction

Proposition: r-Logsob(C) $\implies \|T_t f\|_q \geq \|f\|_p$ for all $f > 0$ and $r' \leq q \leq p \leq r$ if $t \geq \frac{C}{4} \log \frac{1-q}{1-p}$.

Proof Sketch

Assume: $0 < q \leq p \leq r$

$$\text{Let } t(q) = \frac{C}{4} \log \frac{1-q}{1-p}, \quad t(p) = 0, \quad q^2 t'(q) = \frac{Cq^2}{4(q-1)}$$

Let $\psi(q) = \|T_{t(q)} f\|_q$. Note that $\psi(p) = \|f\|_p$

$$\frac{d}{dq} \log \|T_{t(q)} f\|_q = \frac{Ent(f_{t(q)}^q) - q^2 t'(q) \mathcal{E}(f_{t(q)}^{q-1}, f_{t(q)})}{q^2 \mathbb{E} f_{t(q)}^q} \leq 0,$$

since q-logSob(C) holds which follows from r-logSob(C) in turn. The case $r' \leq q \leq p < 0$ follows by duality. The remaining cases follow by taking limits, duality and composition

Log-Sob $\iff$ (Rev) Hyper-contraction

Log Sob $\implies$ Rev-Hyper-Contraction

$\exists C, \| T_{\frac{C}{4}\log\frac{1-q}{1-p}} f \|_q \geq \|f\|_p \; \forall 0 < q < p \leq r \implies \text{r}-logSob(\text{C})$

Remark

Similar results hold for hypercontractivity

Open Problems

Tighter values?

Find tight bounds for simple/general Markov operators

For simple operators we get $\|T_t f\|_q \geq \|f\|_p$ for $q < p \leq 0$ and $t \geq \log \frac{2-q}{2-p}$ and also for $0 \leq q < p < 1$ and $t \geq \log \frac{(1-q)(2-p)}{(1-p)(2-q)}$

Applications?

Applications of strong mixing properties of Markov-chains?

Examples?

Are there examples where r-logSob holds while r'-logSob does not hold for $0 < r < r' < 1$?