[Lecture 4]

0.0.1 Applications of concentration inequalities

Having developed the machinery above, we next apply it to a few concrete problems to give a sense of how to use it. A key lemma which we will use repeatedly is the union bound, which states that if $E_1, \dots, E_n$ are events with probabilities $\pi_1, \dots, \pi_n$, then the probability of $E_1 \cup \cdots \cup E_n$ is at most $\pi_1 + \cdots + \pi_n$. A corollary is that if n events each have probability $\ll 1/n$, then there is a large probability that none of the events occur.

Maximum of sub-Gaussians. Suppose that $X_1, \dots, X_n$ are mean-zero sub-Gaussian with parameter σ, and let $Y = \max_{i=1}^n X_i$. How large is Y? We will show the following:

Lemma 0.1. *The random variable Y is $\mathcal{O}(\sigma\sqrt{\log(n/\delta)})$ with probability $1-\delta$.*

Proof. By the Chernoff bound for sub-Gaussians, we have that $\mathbb{P}[X_i \geq \sigma\sqrt{6\log(n/\delta)}] \leq \exp(-\log(n/\delta)) = \delta/n$. Thus by the union bound, the probability that any of the X_i exceed $\sigma\sqrt{6\log(n/\delta)}$ is at most δ. Thus with probability at least $1-\delta$ we have $Y \leq \sigma\sqrt{6\log(n/\delta)}$, as claimed. ☐

Lemma 0.1 illustrates a typical proof strategy: We first decompose the event we care about as a union of simpler events, then show that each individual event holds with high probability by exploiting independence. As long as the "failure probability" of a single event is much small than the inverse of the number of events, we obtain a meaningful bound. In fact, this strategy can be employed even for an infinite number of events by discretizing to an "ϵ-net", as we will see below:

Eigenvalue of random matrix. Let $X_1, \dots, X_n$ be independent zero-mean sub-Gaussian variables in $\mathbb{R}^d$ with parameter σ, and let $M = \frac{1}{n}\sum_{i=1}^n X_i X_i^\top$. How large is $\|M\|$, the maximum eigenvalue of M? We will show:

Lemma 0.2. *The maximum eigenvalue $\|M\|$ is $\mathcal{O}(\sigma^2 \cdot (1 + d/n + \log(1/\delta)/n))$ with probability $1-\delta$.*

Proof. The maximum eigenvalue can be expressed as

$$\|M\| = \sup_{\|v\|_2 \leq 1} v^\top M v = \sup_{\|v\|_2 \leq 1} \frac{1}{n}\sum_{i=1}^n |\langle X_i, v\rangle|^2. \tag{1}$$

The quantity inside the sup is attractive to analyze because it is an average of independent random variables. Indeed, we have

$$\mathbb{E}[\exp(\frac{n}{\sigma^2} v^\top M v)] = \mathbb{E}[\exp(\sum_{i=1}^n |\langle X_i, v\rangle|^2/\sigma^2)] \tag{2}$$

$$= \prod_{i=1}^n \mathbb{E}[\exp(|\langle X_i, v\rangle|^2/\sigma^2)] \leq 2^n, \tag{3}$$

where the last step follows by sub-Gaussianity if $\langle X_i, v\rangle$. The Chernoff bound then gives $\mathbb{P}[v^\top M v \geq t] \leq 2^n \exp(-nt/\sigma^2)$.

If we were to follow the same strategy as Lemma 0.1, the next step would be to union bound over v. Unfortunately, there are infinitely many v so we cannot do this directly. Fortunately, we can get by with only considering a large but finite number of v; we will construct a finite subset $\mathcal{N}_{1/4}$ of the unit ball such that

$$\sup_{v \in \mathcal{N}_{1/4}} v^\top M v \geq \frac{1}{2} \sup_{\|v\|_2 \leq 1} v^\top M v. \tag{4}$$

Our construction follows Section 5.2.2 of **?**. Let $\mathcal{N}_{1/4}$ be a maximal set of points in the unit ball such that $\|x - y\|_2 \geq 1/4$ for all distinct $x, y \in \mathcal{N}_{1/4}$. We observe that $|\mathcal{N}_{1/4}| \leq 9^d$; this is because the balls of radius $1/8$ around each point in $\mathcal{N}_{1/4}$ are disjoint and contained in a ball of radius $9/8$.

To establish (4), let v maximize $v^\top Mv$ over $\|v\|_2 \le 1$ and let u maximize $v^\top Mv$ over $\mathcal{N}_{1/4}$. Then

$$\begin{aligned} |v^\top Mv - u^\top Mu| &= |v^\top M(v-u) + u^\top M(v-u)| && (5)\\ &\le (\|v\|_2 + \|u\|_2)\|M\|\|v-u\|_2 && (6)\\ &\le 2 \cdot \|M\| \cdot (1/4) = \|M\|/2. && (7) \end{aligned}$$

Since $v^\top Mv = \|M\|$, we obtain $|\|M\| - u^\top Mu| \le \|M\|/2$, whence $u^\top Mu \ge \|M\|/2$, which establishes (4). We are now ready to apply the union bound: Recall that from the Chernoff bound on $v^\top Mv$, we had $\mathbb{P}[v^\top Mv \ge t] \le 2^n \exp(-nt/\sigma^2)$, so

$$\mathbb{P}[\sup_{v \in \mathcal{N}_{1/4}} v^\top Mv \ge t] \le 9^d 2^n \exp(-nt/\sigma^2). \tag{8}$$

Solving for this quantity to equal δ, we obtain

$$t = \frac{\sigma^2}{n} \cdot (n \log(2) + d \log(9) + \log(1/\delta)) = \mathcal{O}(\sigma^2 \cdot (1 + d/n + \log(1/\delta)/n)), \tag{9}$$

as was to be shown. □

VC dimension. Our final example will be important in the following section; it concerns how quickly a family of events with certain geometric structure converges to its expectation. Let $\mathcal{H}$ be a collection of functions $f : \mathcal{X} \to \{0, 1\}$, and define the *VC dimension* $\mathsf{vc}(\mathcal{H})$ to be the maximum d for which there are points $x_1, \ldots, x_d$ such that $(f(x_1), \ldots, f(x_d))$ can take on all 2^d possible values. For instance:

- If $\mathcal{X} = \mathbb{R}$ and $\mathcal{H} = \{\mathbb{I}[x \ge \tau] \mid \tau \in \mathbb{R}\}$ is the family of threshold functions, then $\mathsf{vc}(\mathcal{H}) = 1$.
- If $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{H} = \{\mathbb{I}[\langle x, v\rangle \ge \tau] \mid v \in \mathbb{R}^d, \tau \in \mathbb{R}\}$ is the family of half-spaces, then $\mathsf{vc}(\mathcal{H}) = d + 1$.

Additionally, for a point set $S = \{x_1, \ldots, x_n\}$, let $V_{\mathcal{H}}(S)$ denote the number of distinct values of $(f(x_1), \ldots, f(x_n))$ and $V_{\mathcal{H}}(n) = \max\{V_{\mathcal{H}}(S) \mid |S| = n\}$. Thus the VC dimension is exactly the maximum n such that $V_{\mathcal{H}}(n) = 2^n$.

We will show the following:

Proposition 0.3. *Let $\mathcal{H}$ be a family of functions with $\mathsf{vc}(H) = d$, and let $X_1, \ldots, X_n \sim p$ be i.i.d. random variables over $\mathcal{X}$. For $f : \mathcal{X} \to \{0, 1\}$, let $\nu_n(f) = \frac{1}{n}|\{i \mid f(X_i) = 1\}|$ and let $\nu(f) = p(f(X) = 1)$. Then*

$$\sup_{f \in \mathcal{H}} |\nu_n(f) - \nu(f)| \le \mathcal{O}\Big(\sqrt{\frac{d + \log(1/\delta)}{n}}\Big) \tag{10}$$

with probability $1 - \delta$.

We will prove a weaker result that has a $d \log(n)$ factor instead of d, and which bounds the expected value rather than giving a probability $1 - \delta$ bound. The $\log(1/\delta)$ tail bound follows from *McDiarmid's inequality*, which is a standard result in a probability course but requires tools that would take us too far afield. Removing the $\log(n)$ factor is slightly more involved and uses a tool called *chaining*.

Proof of Proposition 0.3. The importance of the VC dimension for our purposes lies in the Sauer-Shelah lemma:

Lemma 0.4 (Sauer-Shelah)**.** *Let $d = \mathsf{vc}(\mathcal{H})$. Then $V_{\mathcal{H}}(n) \le \sum_{k=0}^{d} \binom{n}{k} \le 2n^d$.*

It is tempting to union bound over the at most $V_{\mathcal{H}}(n)$ distinct values of $(f(X_1), \ldots, f(X_n))$; however, this doesn't work because revealing $X_1, \ldots, X_n$ uses up all of the randomness in the problem and we have no randomness left from which to get a concentration inequality! We will instead have to introduce some new randomness using a technique called *symmetrization*.

Regarding the expectation, let $X'_1, \dots, X'_n$ be independent copies of $X_1, \dots, X_n$ and let $\nu'_n(f)$ denote the version of $\nu_n(f)$ computed with the X'_i. Then we have

$$\mathbb{E}_X[\sup_{f \in \mathcal{H}} |\nu_n(f) - \nu(f)|] \le \mathbb{E}_{X,X'}[\sup_{f \in \mathcal{H}} |\nu_n(f) - \nu'_n(f)|] \quad (11)$$

$$= \frac{1}{n}\mathbb{E}_{X,X'}[\sup_{f \in \mathcal{H}} |\sum_{i=1}^n f(X_i) - f(X'_i)|. \quad (12)$$

We can create our new randomness by noting that since X_i and X'_i are identically distributed, $f(X_i) - f(X'_i)$ has the same distribution as $s_i(f(X_i) - f(X'_i))$, where s_i is a random sign variable that is ± 1 with equal probability. Introducing these variables and continuing the inequality, we thus have

$$\frac{1}{n}\mathbb{E}_{X,X'}[\sup_{f \in \mathcal{H}} |\sum_{i=1}^n f(X_i) - f(X'_i)|] = \frac{1}{n}\mathbb{E}_{X,X',s}[\sup_{f \in \mathcal{H}} |\sum_{i=1}^n s_i(f(X_i) - f(X'_i))|]. \quad (13)$$

We now have enough randomness to exploit the Sauer-Shelah lemma. If we fix X and X', note that the quantities $f(X_i) - f(X'_i)$ take values in $[-1, 1]$ and collectively can take on at most $V_{\mathcal{H}}(n)^2 = \mathcal{O}(n^{2d})$ values. But for fixed X, X', the quantities $s_i(f(X_i) - f(X'_i))$ are independent, zero-mean, bounded random variables and hence for fixed f we have $\mathbb{P}[\sum_i s_i(f(X_i) - f(X'_i)) \ge t] \le \exp(-t^2/9n)$ by Hoeffding's inequality. Union bounding over the $\mathcal{O}(n^{2d})$ effectively distinct f, we obtain

$$\mathbb{P}_s[\sup_{f \in \mathcal{H}} |\sum_i s_i(f(X_i) - f(X'_i))| \ge t \mid X, X'] \le \mathcal{O}(n^{2d}) \exp(-t^2/9n). \quad (14)$$

This is small as long as $t \gg \sqrt{nd \log n}$, so (13) is $\mathcal{O}(\sqrt{d \log n/n})$, as claimed. □

A particular consequence of Proposition 0.3 is the *Dvoretzky-Kiefer-Wolfowitz inequality*:

Proposition 0.5 (DKW inequality)**.** *For a distribution p on $\mathbb{R}$ and i.d.d. samples $X_1, \dots, X_n \sim p$, define the empirical cumulative density function as $F_n(x) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}[X_i \le x]$, and the population cumulative density function as $F(x) = p(X \le x)$. Then $\mathbb{P}[\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \ge t] \le 2e^{-2nt^2}$.*

This follows from applying Proposition 0.3 to the family of threshold functions.