STAT260 Problem Set 2

Due February 23rd by the beginning of class (submit on Gradescope)

Regular problems:

1. Suppose that p^* is $(\sigma, 1/2)$-resilient. Show that $\mathbb{E}_{p^*}[|\langle X - \mu, v\rangle|] \leq C\sigma$ whenever $\|v\|_* \leq 1$, for some absolute constant C.

2. For a norm $\|\cdot\|$ and Orlicz function ψ, define the generalized Orlicz norm $\|X\|_{\psi,\|\cdot\|} = \sup\{\|\langle X, v\rangle\|_\psi \mid \|v\|_* \leq 1\}$. Show that if $\|X - \mu\|_{\psi,\|\cdot\|} \leq \sigma$ for $X \sim p$, and $\epsilon \leq 1/2$, then p is $(2\sigma\epsilon\psi^{-1}(1/\epsilon), \epsilon)$-resilient under the norm $\|\cdot\|$.

3. Prove that if p^* has bounded 4th moments (i.e., $\sup_{\|v\|_2 \leq 1} \mathbb{E}[|\langle x - \mu, v\rangle|^4] \leq \sigma^4$), then $\mathbb{E}[\|x - \mu\|_2^4]^{1/4} \leq C\sigma\sqrt{d}$ for some absolute constant C.

4. Given a κ-approximate relaxation $\mathcal{C}$, and assuming $v^\top \mathsf{Cov}_{p^*}[X]v \leq \sigma^2$ whenever $\|v\|_* \leq 1$, show that the `MinCovNorm` algorithm does indeed produce an estimate $\hat{\mu}$ of $\mu = \mathbb{E}_{p^*}[X]$ such that $\|\hat{\mu} - \mu\| = \mathcal{O}(\sigma\sqrt{\kappa\epsilon})$.

5. Suppose that p^* is sub-Gaussian with parameter σ and $\mathsf{TV}(p^*, \tilde{p}) \leq \epsilon$. Show that given $n \gg \frac{d + \log^2(1/\delta)}{\epsilon^2 \log(1/\epsilon)}$ samples from $\tilde{p}$, we can estimate the mean of p^* with error $\mathcal{O}(\sigma\epsilon\sqrt{\log(1/\epsilon)})$ and probability $1 - \delta$.

   [This can be done by imitating the kth moment proof from class, but for bounding $\|\sum_i \epsilon_i X_i\|_2$ you may wish to use an alternative to Rosenthal; for instance, try obtaining a high probability bound on $\|\sum_i \epsilon_i X_i\|_2$ and then integrating. The $\log^2(1/\delta)$ term is not necessary (we can achieve $\log(1/\delta)$), but is included to give some leeway in the proof.]

Challenge problems (turn in as a separate document typset in LaTeX):

6. Suppose that p^* is $(\sigma, 1/2)$-resilient in the ℓ_2-norm. Show that there is an event E with $\mathbb{P}[E] \geq 1/2$ such that $\|\mathsf{Cov}_{p^*}[X \mid E]\| \leq C\sigma^2$ for some absolute constant C. *[Hint: use the minmax theorem, which states that for $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, if f is convex in x and concave in y then we have*

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} f(x, y),$$

   as long as $\mathcal{X}$ and $\mathcal{Y}$ are convex and at least one of them is compact.]

   While it is not necessary, you are free to assume that p^* is the uniform distribution over samples $\{x_1, \ldots, x_n\}$ if you wish, in order to avoid issues with infinite-dimensional spaces.

7. Suppose we wish to approximate $\sup_{\|v\|_4 \leq 1} v^\top \Sigma v$, where $\|v\|_4 = (|v_1|^4 + \cdots + |v_d|^4)^{1/4}$. As with the ℓ_∞-norm, we can relax this to the convex program

$$\text{maximize } \langle \Sigma, M\rangle \tag{1}$$

$$\text{subject to } M \succeq 0, \sum_i M_{ii}^2 \leq 1. \tag{2}$$

   Show that this provides a $(\pi/2)$-approximate oracle for $\sup_{\|v\|_4 \leq 1} v^\top \Sigma v$, assuming that $\Sigma \succeq 0$.