Examples of non-orthogonal designs

(A) Incomplete block designs

We first consider block designs with $t$ treatments and $b$ blocks of size $k$ with $k < t$. These are called incomplete block designs. Incomplete blocks are used, e.g., when there is a natural limitation on the block size. Or, when there are too many treatments, complete block designs would require large blocks, and it would be more difficult to control the homogeneity within blocks.

When $k < t$, the condition of proportional frequencies cannot be satisfied by the treatment and block factors; therefore $T \ominus G$ has nontrivial projections onto $S_1 = B \ominus G$ and $S_2 = B^\perp$. In this case, there is treatment information in both inter- and intrablock strata. The best linear unbiased estimators of treatment contrasts in $S_1$ and $S_2$ are called inter- and intrablock estimators, respectively.

Since $S_1 = B - G$ and $S_2 = I - B$, by (5.4), (5.5) and (5.6), the normal equations for computing inter- and intrablock estimates of treatment effects are

$$C^1_T \hat{\alpha}^1 = Q^1_T,$$
and
$$C^2_T \hat{\alpha}^2 = Q^2_T,$$
respectively, where

$$C^1_T = X'_T (B - G) X_T, \quad Q^1_T = X'_T (B - G) y,$$
$$C^2_T = X'_T (I - B) X_T, \quad Q^2_T = X'_T (I - B) y.$$

Equation (6.2) is identical to (3.15) in Handout #3. Therefore the intra-block estimator of a treatment contrast under the randomization model is the same as its best linear unbiased estimator under model (3.14) with fixed block effects. By (3.16) and (3.17),

$$C^2_T = \Delta_T - N \Delta_B^1 N'$$
and
$$Q^2_T = S_T - N \Delta_B^1 S_B,$$
where $S_T$ and $S_B$ are the vectors of treatment totals and block totals, respectively. Since the blocks are of the same size $k$, $\Delta_B = k I$. It follows that

$$C^2_T = \Delta_T - \frac{1}{k} N N',$$ 
and
$$Q^2_T = S_T - \frac{1}{k} N S_B,$$

Note that $\Delta_T$ is a $t \times t$ diagonal matrix with the $i$th diagonal entry equal to $r_i$, where $r_i$ is the number of replications of the $i$th treatment. Similarly, one can compute that

$$C^1_T = \frac{1}{k} N N' - \frac{1}{bk} r r',$$
and
$$Q^1_T = \frac{1}{k} N S_B - y_r r,$$
where \( r = (r_1, \ldots, r_t)' \). From Handout #5, we have the following ANOVA table for a general randomized block design:

<table>
<thead>
<tr>
<th>Sources of variation</th>
<th>Sums of Squares</th>
<th>degrees of freedom</th>
<th>Mean square</th>
<th>E(MS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interblock</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments</td>
<td>((\hat{\alpha}_1)'C_j^1\hat{\alpha}_1)</td>
<td>rank ((C_j^1)^{'}C_j^1)</td>
<td>(\frac{1}{\text{rank}(C_j^1)}(\hat{\alpha}_1)'C_j^1\hat{\alpha}_1)</td>
<td>(\xi_1 + \frac{1}{\text{rank}(C_j^1)}\alpha'C_j^1\alpha)</td>
</tr>
<tr>
<td>Residual</td>
<td>By subtraction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\sum_{i=1}^{b}k(y_{ij} - \bar{y}_.)^2)</td>
<td>(b-1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intrablock</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments</td>
<td>((\hat{\alpha}_2)'C_j^2\hat{\alpha}_2)</td>
<td>rank ((C_j^2)^{'}C_j^2)</td>
<td>(\frac{1}{\text{rank}(C_j^2)}(\hat{\alpha}_2)'C_j^2\hat{\alpha}_2)</td>
<td>(\xi_2 + \frac{1}{\text{rank}(C_j^2)}\alpha'C_j^2\alpha)</td>
</tr>
<tr>
<td>Residual</td>
<td>By subtraction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\sum_{i=1}^{b}\sum_{j=1}^{k}(y_{ij} - \bar{y}_.)^2)</td>
<td>(b(k-1))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>(\sum_{i=1}^{b}\sum_{j=1}^{k}(y_{ij} - \bar{y}_.)^2)</td>
<td>(bk-1)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(B) Balanced incomplete block designs

An incomplete block design is said to be binary if \( n_{ij} = 0 \) or 1, \( \forall 1 \leq i \leq t, 1 \leq j \leq b \), where \( n_{ij} \) is the number of times the \( i \)th treatment appears in the \( j \)th block. For a binary design, the \( i \)th diagonal entry of \( NN' \) is \( r_i \) and the \((i,j)\)-th entry is \( \lambda_{ij} \), the number of blocks in which the \( i \)th and \( j \)th treatments appear together. We say that a block design is equi-replicate if all treatments appear the same number of times. Such designs exist only if \( t \mid bk \), in which case we denote \( bk/t \) by \( r \). A balanced incomplete block design (BIBD) with \( t \) treatments and \( b \) blocks of size \( k \) is a binary and equireplicate design in which all pairs of treatments appear together in the same number of blocks, i.e., \( \lambda_{ij} = \lambda \) for a certain constant \( \lambda \). The following is a BIBD with \( b = t = 7 \) and \( k = 3 \):

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 3 & 4 & 5 & 6 & 7 & 1 \\
4 & 5 & 6 & 7 & 1 & 2 & 3
\end{array}
\]

For a BIBD, equations (6.1) and (6.2) are very easy to solve. We have

\[
C_j^2 = \frac{1}{k}[\{r(k - 1) + \lambda\}I_t - \lambda J_t].
\]  

(6.7)
The matrix $\mathbf{Q}_T^2$ has row sum $\frac{1}{k} [r(k - 1) - (t - 1)\lambda]$, which must be zero. Therefore

$$r(k - 1) = (t - 1)\lambda, \quad (6.8)$$

and (6.7) can be written as

$$\mathbf{Q}_T^2 = \frac{1}{k} [t\lambda \mathbf{I}_t - \lambda \mathbf{J}_t]. \quad (6.9)$$

Clearly rank($\mathbf{Q}_T^2$) = $t - 1$. A g-inverse of $\mathbf{Q}_T^2$ is

$$\mathbf{Q}_T^2 = \frac{k}{\lambda} \mathbf{I}_t. \quad (6.10)$$

So a solution of (6.2) is

$$\hat{\alpha}^2 = \frac{k}{\lambda} \mathbf{Q}_T^2, \quad (6.11)$$

From (6.4), the $i$th entry of $\mathbf{Q}_T^2$ is $T_i' \equiv T_i - \frac{b}{k} \sum_{j=1}^{b} n_{ij} B_j$, where $T_i$ is the $i$th treatment total. $T_i'$ is usually called the $i$th adjusted treatment total. Hence the intrablock estimator of any treatment contrast $\mathbf{c}'\alpha$ is

$$\mathbf{c}' \hat{\alpha}^2 = \frac{k}{\lambda} \mathbf{c}' \mathbf{Q}_T^2 = \frac{k}{\lambda} \sum_{i=1}^{t} c_i T_i', \quad (6.12)$$

with

$$\text{var}(\mathbf{c}' \hat{\alpha}^2) = \xi_2 \mathbf{c}'(\mathbf{C}_T^2)^{-1}\mathbf{c} = \xi_2 \frac{k}{\lambda} \mathbf{c}' \mathbf{c}. \quad (6.13)$$

So all normalized contrasts are estimated with the same precision. In particular, each pairwise comparison is estimated with variance

$$\text{var}(\hat{\alpha}_i^2 - \hat{\alpha}_j^2) = 2 \frac{k}{\lambda} \xi_2.$$

From (6.11), the treatment sum of squares in the intrablock stratum is

$$(\hat{\alpha}^2)' \mathbf{Q}_T^2 = \frac{k}{\lambda} (\mathbf{Q}_T^2)' \mathbf{Q}_T^2 = \frac{k}{\lambda} \sum_{i=1}^{t} (T_i')^2.$$

For estimation in the interblock stratum, we have, from (6.5),

$$\mathbf{Q}_T^1 = \frac{1}{k} [(r - \lambda) \mathbf{I}_t + \lambda \mathbf{J}_t] - \frac{r^2}{tk} \mathbf{J}_t. \quad (6.14)$$

This matrix also has rank $t - 1$, with $\frac{k}{r-\lambda} \mathbf{I}_t$ as a g-inverse. Therefore a solution of (6.1) is

$$\hat{\alpha}^1 = \frac{k}{r-\lambda} \mathbf{Q}_T^1, \quad (6.15)$$
The intrablock estimator of any treatment contrast \( c'\alpha \) is

\[
c'\hat{\alpha} = \frac{k}{r-\lambda} c'Q_T^1
\]

\[
= \frac{k}{r-\lambda} c'(\frac{1}{r}NS_B - y_r)
\]

[by (6.6)]

\[
= \frac{k}{r-\lambda} c'(\frac{1}{r}NS_B - y_r1_t)
\]

[since \( r_1 = \ldots = r_t = r \)]

\[
= \frac{1}{r-\lambda} c'NS_B
\]

(since \( c'1_t = 0 \))

\[
= \frac{1}{r-\lambda} \sum_{i=1}^t c_i(\sum_{j=1}^b n_{ij}B_j),
\]

with

\[
\text{var}(c'\hat{\alpha}) = \xi_1 c'(C_T^1)^-c = \xi_1 \frac{k}{r-\lambda} c'c.
\]

(6.15)

Again all normalized contrasts are estimated with the same precision. From (6.14), the treatment sum of squares in the interblock stratum is

\[
(\hat{\alpha}^1)'Q_T^1 = \frac{k}{r-\lambda}(Q_T^1)'Q_T^1.
\]

We have the following ANOVA table for a BIBD:

<table>
<thead>
<tr>
<th>Sources of variation</th>
<th>Sums of Squares</th>
<th>degrees of freedom</th>
<th>Mean square</th>
<th>E(MS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interblock</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments</td>
<td>( \frac{k}{r-\lambda}(Q_T^1)'Q_T^1 )</td>
<td>( t - 1 )</td>
<td>( \frac{1}{t-1} \frac{k}{r-\lambda}(Q_T^1)'Q_T^1 )</td>
<td>( \xi_1 + \frac{1}{t-1} \alpha'C_T^1\alpha )</td>
</tr>
<tr>
<td>Residual</td>
<td>By subtraction</td>
<td>:</td>
<td>( \xi_1 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \sum_{i=1}^b k(y_{i.} - y_.)^2 )</td>
<td>( b - 1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intrablock</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments</td>
<td>( \frac{k}{r-\lambda} \sum_{i=1}^t (T_i')^2 )</td>
<td>( t - 1 )</td>
<td>( \frac{1}{t-1} \frac{k}{r-\lambda} \sum_{i=1}^t (T_i')^2 )</td>
<td>( \xi_2 + \frac{1}{t-1} \alpha'C_T^2\alpha )</td>
</tr>
<tr>
<td>Residual</td>
<td>By subtraction</td>
<td>:</td>
<td>( \xi_2 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \sum_{i=1}^b \sum_{j=1}^k (y_{ij} - y_.)^2 )</td>
<td>( b(k - 1) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>( \sum_{i=1}^b \sum_{j=1}^k (y_{ij} - y_.)^2 )</td>
<td>( bk - 1 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Now compare a BIBD with a hypothetical randomized complete block design in which each treatment appears \( r \) times. Under such a design, \( \text{var}(c\hat{\alpha}) = c'\xi_2/r \); or since it would involve a different grouping of the units, let's write this variance as \( c'\xi_2'/r \). Then

\[
\frac{\text{var}(c'\hat{\alpha})}{\text{var}(c'\hat{\alpha}')} \text{ for a randomized complete block design} = \frac{t\lambda}{rk} \cdot \frac{\xi_2}{\xi_2'},
\]

The ratio \( t\lambda/rk \), denoted by \( e \), is called the efficiency factor (or more precisely, the efficiency factor in the intrablock stratum) of the BIBD. Similarly,

\[
\frac{\text{var}(c'\hat{\alpha})}{\text{var}(c'\hat{\alpha}')} \text{ for a randomized complete block design} = \frac{r-\lambda}{rk} \cdot \frac{\xi_2'}{\xi_1'}.
\]

The factor \( (r-\lambda)/rk \) may be called the efficiency factor in the interblock stratum. We have \( t\lambda/rk + (r-\lambda)/rk = [(t-1)\lambda + r]/rk = [(k-1)r + r]/rk = 1 \) (The next-to-last equality follows from (6.8)). Therefore the efficiency factor in the interblock stratum is simply \( 1 - e \), and we have \( 0 < e < 1 \). So for a BIBD,

\[
\frac{\text{var}(c'\hat{\alpha}^2)}{\text{var}(c'\hat{\alpha})} = \frac{1-e}{e} \cdot \frac{\xi_2}{\xi_1}.
\]

By (6.8), the efficiency factor can also be expressed as

\[
e = \frac{t\lambda}{rk} = \frac{t(k-1)}{(t-1)k}.
\]

(C) Recovery of interblock information

The interblock and intrablock estimators are uncorrelated. [In general, estimators in different strata are uncorrelated: \( i \neq j \Rightarrow \text{cov}(a'S_i\hat{y}, b'S_j\hat{y}) = \text{cov}(a'S_i, b'S_j) = 0 \).] Suppose \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are two uncorrelated unbiased estimators of a certain parameter \( \theta \). Then it is easy to see that among unbiased estimators of the form \( a_1\hat{\theta}_1 + a_2\hat{\theta}_2 \), \( \text{var}(a_1\hat{\theta}_1 + a_2\hat{\theta}_2) \) is minimized by

\[
a_i = \frac{1}{\text{var}(\hat{\theta}_i) + \text{var}(\hat{\theta}_2)} \cdot i = 1, 2.
\]

If \( \xi_1 \) and \( \xi_2 \) were known, then the best linear combination of the inter- and intrablock estimators of a treatment contrast would be

\[
\frac{\frac{t\lambda}{\xi_2}c'\hat{\alpha}^2 + \frac{r-\lambda}{\xi_1}c'\hat{\alpha}'}{\frac{t\lambda}{\xi_2} + \frac{r-\lambda}{\xi_1}}.
\]

It can be shown that this is the BLUE of \( c'\hat{\alpha} \) (but it may not be true for other block designs). Since \( \xi_1 \) and \( \xi_2 \) are unknown, they need to be estimated.
(D) Non-orthogonal row-column designs

Suppose there are $t$ treatments and the block structure is $r \times c$. We have $S_1 = R - G$, $S_2 = C - G$, and $S_3 = I - R - C + G$. By (5.4), (5.5) and (5.6), the normal equations for computing estimates of treatment effects in the three strata are

$$C_T^1 \alpha = Q_T^1,$$

$$C_T^2 \alpha = Q_T^2,$$

and

$$C_T^3 \alpha = Q_T^3,$$

respectively, where

$$C_T^1 = X_T^T (R - G) X_T, Q_T^1 = X_T^T (R - G) y,$$

$$C_T^2 = X_T^T (C - G) X_T, Q_T^2 = X_T^T (C - G) y,$$

$$C_T^3 = X_T^T (I - R - C + G) X_T, Q_T^3 = X_T^T (I - R - C + G) y.$$ 

Equations (6.17) and (6.18) are the same as the equation for calculating the interblock estimators except that $B$ is replaced by $R$ and $C$, respectively. Therefore, the inter-row (respectively, inter-column) estimators are the same as the interblock estimators when the rows (respectively, columns) are considered as blocks, i.e.,

$$C_T^1 = \frac{1}{c} N_R N_R' - \frac{1}{rc} rr', Q_T^1 = \frac{1}{c} N_R S_R - y_r,$$

and

$$C_T^2 = \frac{1}{r} N_C N_C' - \frac{1}{rc} rr', Q_T^2 = \frac{1}{r} N_C S_C - y_r,$$

where $N_R$ is the treatment-row incidence matrix, $N_C$ is the treatment-column incidence matrix, $S_R$ is the vector of row totals, and $S_C$ is the vector of column totals. The ANOVA in the inter-row and inter-column strata are the same as that in the inter-block stratum given in this handout. Therefore we shall now concentrate on the calculation of the estimates in $S_3$. Similar to what we saw earlier for block designs, the estimators of treatment contrasts in stratum $S_3$ are the same as the best linear unbiased (least squares) estimators under the usual additive fixed effects model:

$$y_{ij} = \mu + \alpha t(i,j) + \beta_i + \gamma_j + \epsilon_{ij},$$

where $y_{ij}$ is the observation at the $i$th row and $j$th column, $t(i,j)$ is the treatment assigned there, the $\alpha$, $\beta$ and $\gamma$'s are unknown constants representing treatment, row and column effects, $E(\epsilon_{ij}) = 0$, and all the observations are uncorrelated with constant variance.

By direct computation,

$$C_T^3 = \Delta_T - \frac{1}{c} N_R N_R' - \frac{1}{r} N_C N_C' + \frac{1}{rc} rr',$$

and
\[ Q_T^3 = S_T - \frac{1}{r} N_R S_R - \frac{1}{c} N_C S_C + y_i r. \]

(E) Row-column designs with \( N_R = mJ \).

Suppose we have a row-column design with \( N_R = mJ \), i.e., each treatment appears \( m \) times in each row. Then the condition of proportional frequencies is satisfied by the treatment and row factors. It follows that \( T \perp G \perp R \perp G \); so there is no treatment information in \( S_1 \). We already know that the inter-column estimators are the same as the inter-block estimators with the columns considered as blocks. For the estimators in \( S_3 \), since \( X_T'(R - G) = 0 \), we have
\[ C_T^3 = X_T'(I - R - C + G)X_T = X_T'(I - C)X_T \quad (6.20) \]

and
\[ Q_T^3 = X_T'(I - R - C + G)y = X_T'(I - C)y. \]

This show that the estimators in \( S_3 \) are the same as the \textit{intra-block} estimators with the columns considered as blocks. Therefore we have the following ANOVA table for a row-column design with \( N_R = mJ \):

<table>
<thead>
<tr>
<th>Sources of variation</th>
<th>Sums of Squares</th>
<th>d. f.</th>
<th>Mean square</th>
<th>E(MS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>rows</td>
<td>( \sum c(y_i - y_i)^2 )</td>
<td>( r - 1 )</td>
<td>( \frac{1}{r-1} \sum c(y_i - y_i)^2 )</td>
<td>( \xi_1 )</td>
</tr>
<tr>
<td>columns</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments</td>
<td>((\hat{\alpha}^2)'C_T^2\hat{\alpha}^2)</td>
<td>rank ((C_T^2))</td>
<td>( \frac{1}{\text{rank}(C_T^2)}(\hat{\alpha}^2)'C_T^2\hat{\alpha}^2 )</td>
<td>( \xi_2 + \frac{1}{\text{rank}(C_T^2)}\alpha'C_T^2\alpha )</td>
</tr>
<tr>
<td>Residual</td>
<td>By subtraction</td>
<td>:</td>
<td></td>
<td>( \xi_2 )</td>
</tr>
<tr>
<td></td>
<td>( \sum_j c(y_j - y_j)^2 )</td>
<td>( c - 1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>row.column</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments</td>
<td>((\hat{\alpha}^3)'C_T^3\hat{\alpha}^3)</td>
<td>rank ((C_T^3))</td>
<td>( \frac{1}{\text{rank}(C_T^3)}(\hat{\alpha}^3)'C_T^3\hat{\alpha}^3 )</td>
<td>( \xi_3 + \frac{1}{\text{rank}(C_T^3)}\alpha'C_T^3\alpha )</td>
</tr>
<tr>
<td>Residual</td>
<td>By subtraction</td>
<td>:</td>
<td></td>
<td>( \xi_3 )</td>
</tr>
<tr>
<td></td>
<td>( \sum_i \sum_j c(y_{ij} - y_i - y_j + y_.)^2 )</td>
<td>( (r-1)(c-1) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>( \sum_i \sum_j c(y_{ij} - y_i)^2 )</td>
<td>( rc - 1 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If \( N_R = J \) and \( N_C \) is the incidence matrix of a BIBD, then it is called a \textit{Younen Square}. 

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