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0

DRB notation.

Data $\quad Y(t),\ t = 0, \ldots, T-1$

DFT

$$d_Y^T(\lambda) = \sum_{t=0}^{T-1} e^{-i\lambda t}\, Y(t), \quad -\infty < \lambda < \infty$$

Periodogram

$$I_{YY}^T(\lambda) = \frac{1}{2\pi T} \left| d_Y^T(\lambda) \right|^2$$

Power spectrum

$$f_{YY}(\lambda)$$

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Likelihood analysis. Whittle approach

Suppose $f_{YY}(\lambda|\theta)$ θ: finite-d parameter

Let's act as if the periodogram values are independent exponentials

Corresponding likelihood

$$\prod_s \frac{1}{f_{YY}\left(\frac{2\pi s}{T}|\theta\right)} \exp\left\{-\sum_s I_{YY}^T\left(\frac{2\pi s}{T}\right) \Big/ f_{YY}\left(\frac{2\pi s}{T}|\theta\right)\right\}$$

− log likelihood

$$Q^T(\theta) = \frac{2\pi}{T}\sum_s \left\{ I_{YY}^T\left(\frac{2\pi s}{T}\right) \Big/ f_{YY}\left(\frac{2\pi s}{T}|\theta\right) + \log f_{YY}\left(\frac{2\pi s}{T}|\theta\right)\right\}$$

$$\xrightarrow{P} Q(\theta) = \int_0^{2\pi} \left\{ \frac{f_{YY}(\lambda|\theta_0)}{f_{YY}(\lambda|\theta)} + \log f_{YY}(\lambda|\theta)\right\} d\lambda$$

cp. $J^T(A)$

A maximum at θ_0
- suppose unique
θ identifiable

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Set $\Lambda^T(\theta) = \frac{\partial Q^T}{\partial \theta} = -\frac{2\pi}{T} \sum_s \{ I^T - f \} \frac{1}{f} \frac{\partial \log f}{\partial \theta}$

$\xrightarrow{P} \Lambda(\theta)$ cp. $J^T(A)$

$\Lambda(\theta_0) = 0$

$\Lambda^T(\hat{\theta}) = 0$ via iterative routines, initial values

Consistency

Asymptotic normality

$\Lambda^T(\hat{\theta}) \simeq \Lambda^T(\theta_0) + \frac{\partial \Lambda^T(\theta_0)}{\partial \theta_0}(\hat{\theta} - \theta_0)$

$\sqrt{T}(\hat{\theta} - \theta_0) \sim -\left[\frac{\partial \Lambda^T(\theta_0)}{\partial \theta_0}\right]^{-1} \sqrt{T}\Lambda^T(\theta_0)$

$\sqrt{T}\Lambda^T(\theta_0) \xrightarrow{P} N(0, \underset{\sim}{C})$

$\frac{\partial \Lambda^T(\theta_0)}{\partial \theta_0} \longrightarrow A$

$\hat{\theta} \sim N(\theta_0, \frac{1}{T}\underset{\sim}{A}^{-1}\underset{\sim}{C}\underset{\sim}{A}^{-1})$

Dzhaparidze (1986) book asymp efficient, asymp $\equiv$ mle

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In Gaussian estimation $\underset{\sim}{A} = [A_{jk}]$

$$A_{jk} = \frac{1}{2}\int_0^{2\pi} \left\{\frac{\partial}{\partial\theta_j}\log f_{YY}(\lambda|\theta)\right\}\left\{\frac{\partial}{\partial\theta_k}\log f_{YY}(\lambda|\theta)\right\} d\lambda$$

cf. mle

$$\underset{\sim}{C} = 2\pi(\underset{\sim}{A} + \underset{\sim}{B})$$

$$B_{jk} = \frac{1}{4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{f_{YY}(\alpha|\theta)}\frac{\partial \log f_{YY}(\alpha|\theta)}{\partial\theta_j}\right\}\left\{\frac{1}{f_{YY}(\beta|\theta)}\frac{\partial \log f_{YY}(\beta|\theta)}{\partial\theta_k}\right\}$$

$$f_{YYYY}(\alpha, \beta, -\beta)\, d\alpha\, d\beta$$

Notes.

If $Y(.)$ Gaussian, $f_{YYYY} \equiv 0$

get $\frac{2\pi}{T}\underset{\sim}{A}^{-1}$

Can use for: CI's
tests
inferences

Particular cases. Fitting ARMA's

1. AR(p)

$$\underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{R} & 0 \\ 0 & 1/2\sigma^4 \end{bmatrix}$$ need $\hat{\sigma}^2$

with

$$\underset{\sim}{R} = [c_{YY}(j-k)] \quad \left\{ = \int e^{i(j-k)\lambda} |A(\lambda)|^2 d\lambda \right\}$$

OLS is efficient

2. MA(q)

$$\underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{S} & 0 \\ 0 & 1/2\sigma^4 \end{bmatrix}$$

$$\underset{\sim}{S} = \frac{\sigma^2}{2\pi} \left[\int_{-\pi}^{\pi} \frac{e^{i(j-k)\lambda}}{|B(\lambda)|^2} d\lambda \right]$$

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3. ARMA(p, q)

$$\underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{R} & -\overline{\underset{\sim}{T}} & \underset{\sim}{0} \\ -\underset{\sim}{T} & \underset{\sim}{S} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & 1/2\sigma^4 \end{bmatrix}$$

$$\underset{\sim}{T} = \frac{\sigma^2}{2\pi}\left[\int_{-\pi}^{\pi} \frac{e^{i(j-k)\lambda}}{A(\lambda)\,\overline{B(\lambda)}}\,d\lambda\right]$$

$$\underset{\sim}{S} = \frac{\sigma^2}{2\pi}\left[\int_{-\pi}^{\pi} \frac{e^{i(j-k)\lambda}}{|B(\lambda)|^2}\,d\lambda\right]$$

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<u>Case with a signal</u>

$$Y(t) = S(t|\theta) + \epsilon(t)$$

$\epsilon(t)$: mean 0
spectrum $f_{\epsilon\epsilon}(\lambda|\theta)$

Periodogram ordinates $I^T_{\epsilon\epsilon}(\frac{2\pi s}{T})$ approx independent exponentials mean $f_{\epsilon\epsilon}(\frac{2\pi s}{T})$

Likelihood

$$L(\theta) = \prod_s \frac{1}{f_{\epsilon\epsilon}(\frac{2\pi s}{T}|\theta)} \exp\left\{-\sum_s I^T_{\epsilon\epsilon}\left(\frac{2\pi s}{T}\right) / f_{\epsilon\epsilon}\left(\frac{2\pi s}{T}|\theta\right)\right\}$$

$$d^T_Y(\lambda) = d^T_S(\lambda|\theta) + d^T_\epsilon(\lambda)$$

$$I^T_{\epsilon\epsilon}(\lambda) = \frac{1}{2\pi T}\left|d^T_Y(\lambda) - d^T_S(\lambda|\theta)\right|^2$$

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<u>Vector case</u> $\underset{\sim}{Y}(t)$, spectral density matrix $\underset{\sim}{f}_{YY}(\lambda|\theta)$

$$\underset{\sim}{d}_Y^T(\lambda) = \sum_{t=0}^{T-1} e^{-i\lambda t}\, \underset{\sim}{Y}(t)$$

$$\underset{\sim}{I}_{YY}^T(\lambda) = \frac{1}{2\pi T}\, \underset{\sim}{d}_Y^T(\lambda)\, \overline{\underset{\sim}{d}_Y^T(\lambda)}^{\tau}$$

$$\underset{\sim}{I}_{YY}^T\left(\frac{2\pi s}{T}\right) \quad \text{approx} \quad IW_r^c\left(1, \underset{\sim}{f}_{YY}\left(\frac{2\pi s}{T}\Big|\theta\right)\right)$$

Likelihood $\quad L(\theta) =$

$$\prod_s \left[\pi^{r(r-1)/2}\right]^{-1} \left|\underset{\sim}{f}_{YY}\left(\frac{2\pi s}{T}\Big|\theta\right)\right|^{-1} \left|\underset{\sim}{I}_{YY}^T\left(\frac{2\pi s}{T}\right)\right|^{1-r}$$
$$\times \exp\left\{-\mathrm{tr}\left[\underset{\sim}{f}_{YY}\left(\frac{2\pi s}{T}\Big|\theta\right)^{-1} \underset{\sim}{I}_{YY}^T\left(\frac{2\pi s}{T}\right)\right]\right\}$$

$$|\underset{\sim}{\Sigma}| = \operatorname{Det} \underset{\sim}{\Sigma}$$