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Spectral representation. $\{X_t\}$ covariance stationary $\exists Z(\lambda) \rightarrow$

$$X_t = \int_{-\pi}^{\pi} e^{it\lambda} dZ(\lambda), \quad t=0, \pm 1, \dots$$

with $\overline{Z(\lambda)} = Z(-\lambda),$

$$\text{cov} \{dZ(\lambda), dZ(\mu)\} = \eta(\lambda) f(\lambda) d\lambda d\mu$$

$$[\eta(\lambda) dF(\lambda) d\mu]$$

Cramér

$$\eta(\lambda) = \sum_j \delta(\lambda + 2\pi j)$$

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Filtering

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k X_{t-k}$$

$$\Psi(e^{-i\lambda}) = \sum_{j=-\infty}^{\infty} \psi_j e^{-i\lambda j}$$

$$dZ_Y(\lambda) = \Psi(e^{-i\lambda}) dZ_X(\lambda)$$

$$f_{YY}(\lambda) = |\Psi(e^{-i\lambda})|^2 f_{XX}(\lambda)$$

e.g. b.p. filter

$$Y(\lambda) = \left(\int_{-\lambda_0 - \Delta}^{-\lambda_0 + \Delta} + \int_{\lambda_0 - \Delta}^{\lambda_0 + \Delta} \right) e^{i\lambda} dZ_X(\lambda)$$

\exists algebra

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Problem 3.5. $\{W_t\} \sim WN(0, \sigma_w^2)$

$\{X_t\} \sim \text{ARMA}(p, q)$

$$r = \max(p, q)$$

Consider $Y_t = X_t + W_t$

$$f_{YY}(\lambda) = f_{XX}(\lambda) + f_{WW}(\lambda)$$

$$= \frac{\sigma^2 |\Theta(e^{-i\lambda})|^2}{2\pi |\varphi_p(e^{-i\lambda})|^2} + \frac{\sigma_w^2}{2\pi}$$

$$= \frac{1}{2\pi} \frac{\sigma^2 |\Theta(e^{-i\lambda})|^2 + \sigma_w^2 |\varphi_p(e^{-i\lambda})|^2}{|\varphi_p(e^{-i\lambda})|^2}$$

$\sim \text{ARMA}(p, r)$

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Aliasing

Continuous time

$$X_t = \int_{-\infty}^{\infty} e^{it\lambda} d\tilde{Z}(\lambda), \quad -\infty < t < \infty$$

Discrete observations

$$X_t = \int_{-\pi}^{\pi} e^{it\lambda} dZ(\lambda), \quad \lambda = 0, \pm 1, \pm 2, \dots$$

$$X_t = \sum_j \int_{2\pi j}^{2\pi(j+1)} e^{it\lambda} d\tilde{Z}(\lambda)$$

$$= \int_0^{2\pi} e^{it\lambda} \sum_j d\tilde{Z}(\lambda - 2\pi j)$$

$$dZ(\lambda) = \sum_j d\tilde{Z}(\lambda - 2\pi j)$$

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$$f(\lambda) = \sum_j g(\lambda - 2\pi j)$$

$g(\cdot)$ cts spectrum

Aliases $\left\{ \begin{array}{l} \lambda + 2\pi j \\ -\lambda + 2\pi j \end{array}, j = 0, \pm 1, \dots \right\}$

Anti aliasing filter

$$t = 0, \pm h, \pm 2h, \dots$$

$$\left\{ \begin{array}{l} \lambda + \frac{2\pi j}{h} \\ \lambda - \frac{2\pi j}{h} \end{array}, j = 0, \pm 1, \dots \right\}$$

$$\pi/h$$

Nyquist / folding frequency

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E.g. $X(t) = R \cos(\omega t + \varphi)$ $\varphi \sim U(-\pi, \pi)$

$$Y(h) = \frac{1}{2} R^2 \cos \omega h$$

continuous case

$$\frac{1}{4} R^2 [\delta(\lambda - \omega) + \delta(\lambda + \omega)]$$

discrete case

$$\frac{1}{4} R^2 [\pi(\lambda - \omega) + \pi(\lambda + \omega)]$$

E.g. sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{t=0}^{n-1} X_t = \frac{1}{n} \int_{-\pi}^{\pi} \left(\sum_{t=0}^{n-1} e^{it\lambda} \right) dZ(\lambda)$$

$$\sum_{t=0}^{n-1} e^{it\lambda} = \Delta^n(\lambda)$$

$$\text{var } \bar{X}_n = \frac{1}{n^2} \int_{-\pi}^{\pi} |\Delta^n(\lambda)|^2 f(\lambda) d\lambda$$

$$= \frac{1}{n^2} \int_{-\pi}^{\pi} \left(\frac{\sin n\lambda/2}{\sin \lambda/2} \right)^2 f(\lambda) d\lambda$$

$$\approx \frac{2\pi}{n} f(0)$$

DFT.

$$A_k = \frac{1}{\sqrt{n}} \sum_{t=1}^n X_t e^{-it\omega_k}$$