7 April 04

Association of (stationary) time series

Measuring and asking if it is due to chance.

Some characteristics of dependence:

$$E(XY) \neq E(X)E(Y)$$

$$E(Y|X) = g(X)$$

$$X = g(\xi),\ Y = h(\xi) \qquad \xi:\ \text{latent}$$

$$f_{XY}(x,y) \neq f_X(x) f_Y(y) \quad \text{for some } (x,y)$$

Mutual information, $I_{XY} \neq 0$

$$I_{XY} = \underset{X,Y}{E}\left\{ \log \frac{f_{XY}(X,Y)}{f_X(X) f_Y(Y)} \right\}$$

For a test, need to keep alternatives in mind.

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Time series $(X(t), Y(t))$, $t = 0, \pm 1, \pm 2, \ldots$

stationary, mixing

1. <u>Time-side statistic</u>

$$\operatorname{cov}\{X(t+u), Y(t)\} = c_{XY}(u)$$

Is it 0 at lag u? (Alternative lag u dependence

$$c^T_{XY}(u) = \frac{1}{T} \sum_{0 \le t, t+u \le T-1} [X(t+u) - \bar{X}][Y(t+u) - \bar{Y}]$$

∃ asymptotic distribution
DRB book (7.6.11)

$$\frac{T}{2\pi} \operatorname{var} c^T_{XY}(u) \sim \int_0^{2\pi} f_{XX}(\alpha) f_{YY}(\alpha)\, d\alpha$$

$$+ \int_0^{2\pi} e^{-i2u\alpha} |f_{XY}(\alpha)|^2 d\alpha$$

$$+ \int^{2\pi} \int^{2\pi} e^{i(u\alpha - u\beta)} f_{XYXY}(\alpha, -\alpha, -\beta)\, d\alpha\, d\beta$$

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Crude result for unrelated <u>linear</u> processes

$$\operatorname{var}\left\{\frac{c^T_{XY}(u)}{\sqrt{c^T_{XX}(0)\,c^T_{YY}(0)}}\right\} \sim \frac{1}{T}$$

(but need to consider behavior under alternative)

Other ways to get var $c^T_{XX}(0)$:

1. Split sample
2. Via spectrum of product $Z(t) = X(t)Y(t)$
3. Bootstraps

<u>Note</u>. Could have $c_{XY}(0) = 0$, but X and Y dependent.

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2. Frequency-side statistic.

Coherence. A measure of linear time invariant association as a function of frequency.

$$|R_{YX}(\lambda)|^2 = \frac{|f_{XY}(\lambda)|^2}{f_{XX}(\lambda) f_{YY}(\lambda)} = |R_{XY}(\lambda)|^2$$

if denom $\neq 0$

$= 0$ if denom $= 0$

$$0 \leq |R_{XY}(\lambda)|^2 \leq 1$$

Proof. Consider

$$d_X^T(\lambda) = \sum_{t=0}^{T-1} e^{-i\lambda t} X(t) \; , \quad d_Y^T(\lambda) =$$

$$\text{var}\, d_X^T(\lambda) = \int |\Delta^T(\lambda - \alpha)|^2 f_{XX}(\alpha) d\alpha$$

from Cramér

$\sim 2\pi T f_{XX}(\lambda)$ for large T

$$\text{cov}\{d_X^T(\lambda), d_Y^T(\lambda)\} = \int |\Delta^T(\lambda - \alpha)|^2 f_{XY}(\alpha) d\alpha$$

$$\sim 2\pi T f_{XY}(\lambda)$$

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But $|\mathrm{cov}\{X, Y\}|^2 \le \mathrm{var}X \; \mathrm{var}Y$
Schwarz

So

$$|\mathrm{corr}\{d_X^T(\lambda), d_Y^T(\lambda)\}|^2 \le 1$$

Take limit as $T \to \infty$.

Can ~~estimate~~ coherence via

$$\frac{|f_{XY}^T(\lambda)|^2}{f_{XX}^T(\lambda) \, f_{YY}^T(\lambda)}$$

Density of large sample distribution

$$(1 - |R|^2)^n \; {}_2F_1(n, n; 1, |R|^2 |\hat{R}|^2) \frac{\Gamma(n)}{\Gamma(n-1)\Gamma(1)} (1 - |\hat{R}|^2)^{n-2}$$

if n periodogram ordinates averaged to obtain $\hat{f}$'s

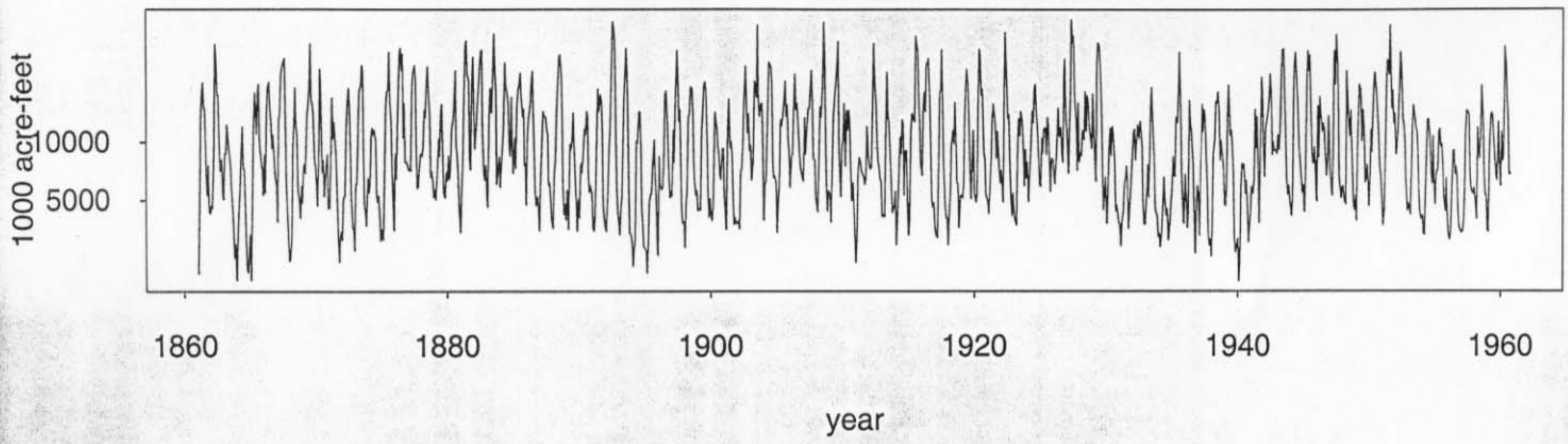
If $|R|^2 = 0$, then

$$E|\hat{R}|^2 \sim \frac{1}{n}$$

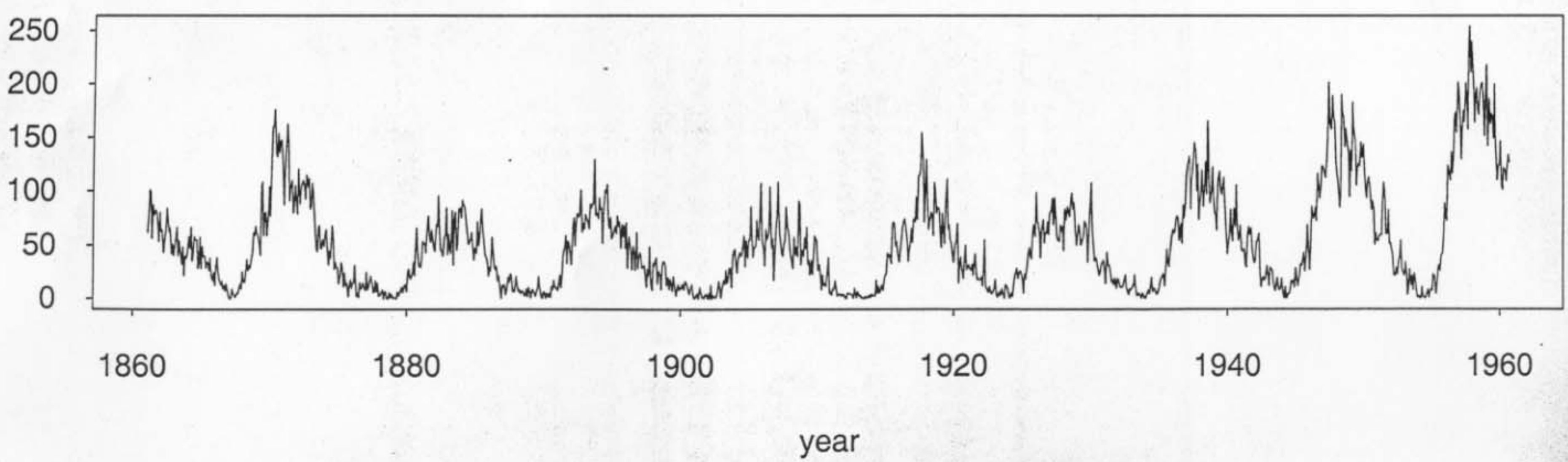
$100\alpha\%$ point $\quad 1 - (1-\alpha)^{1/(n-1)}$

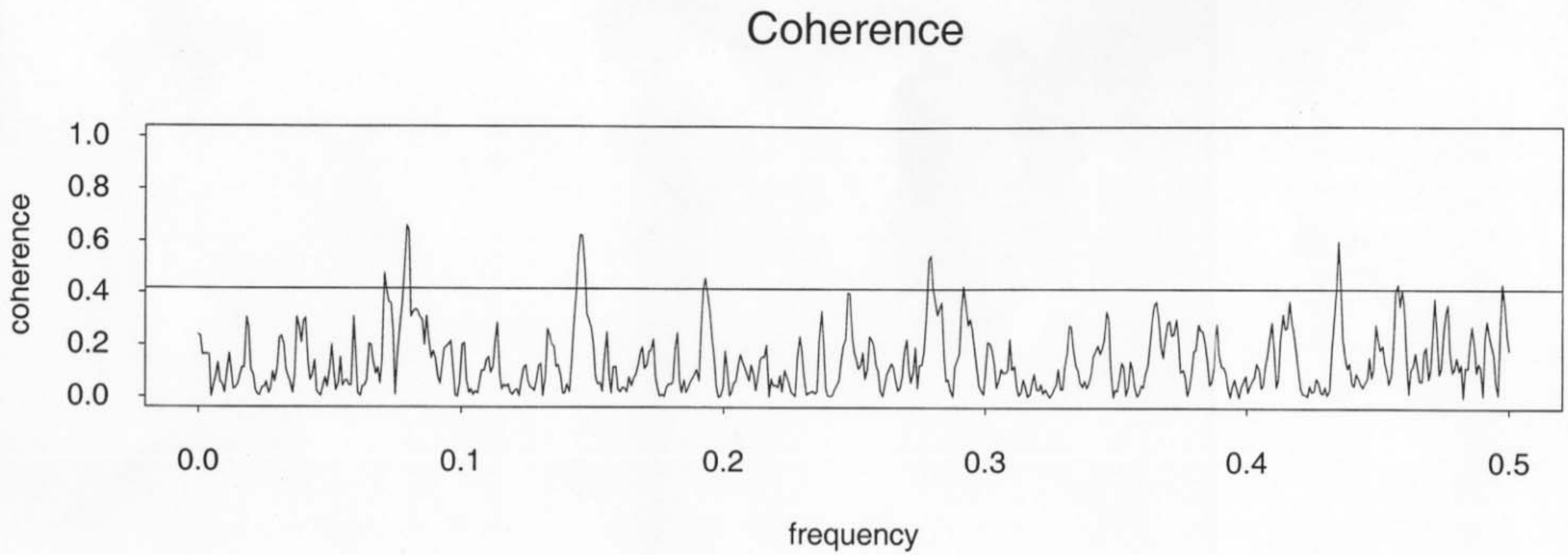
Degrees of freedom are $2n$

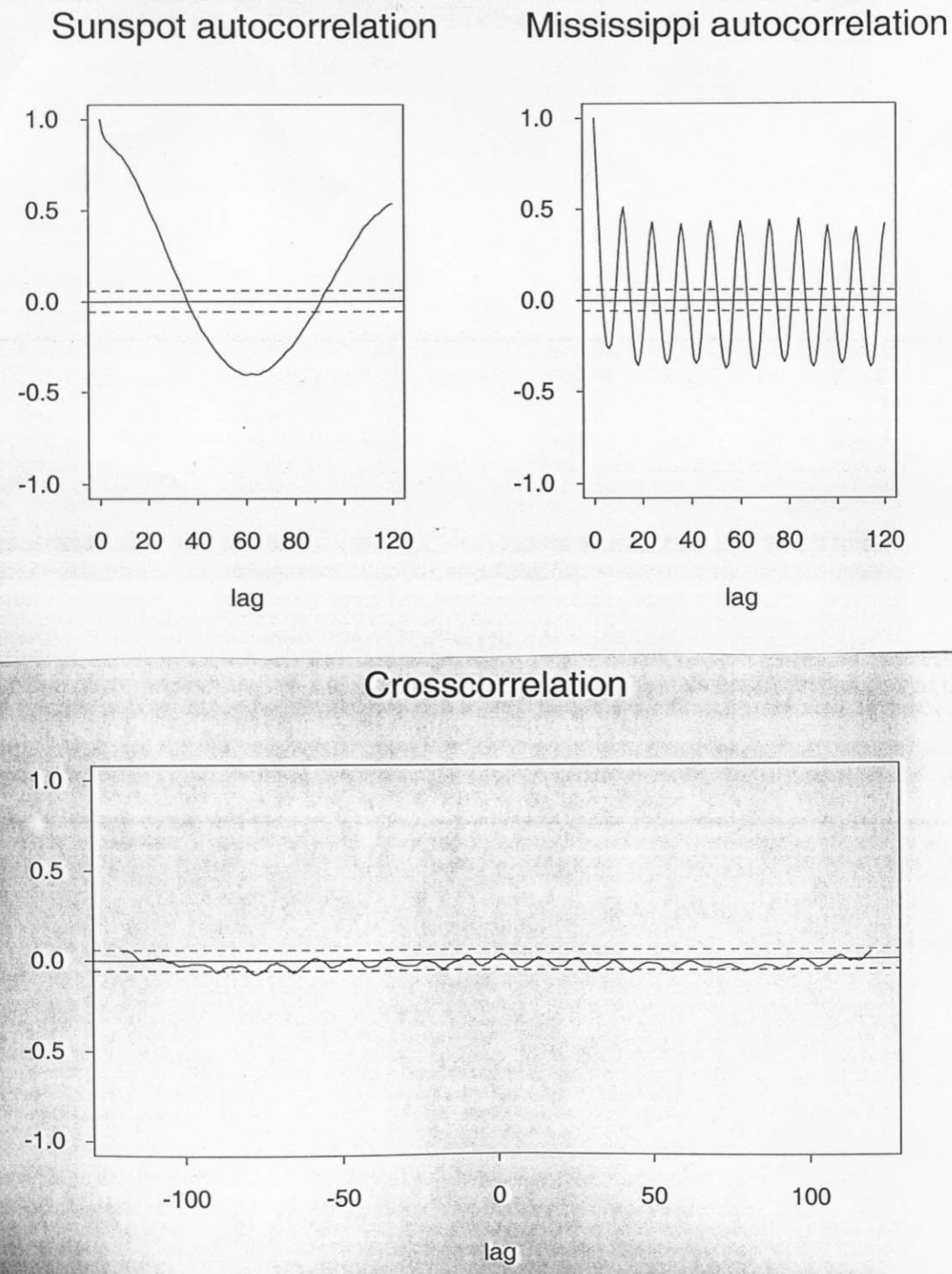
Monthly Mississippi river runoff, 1861-1960



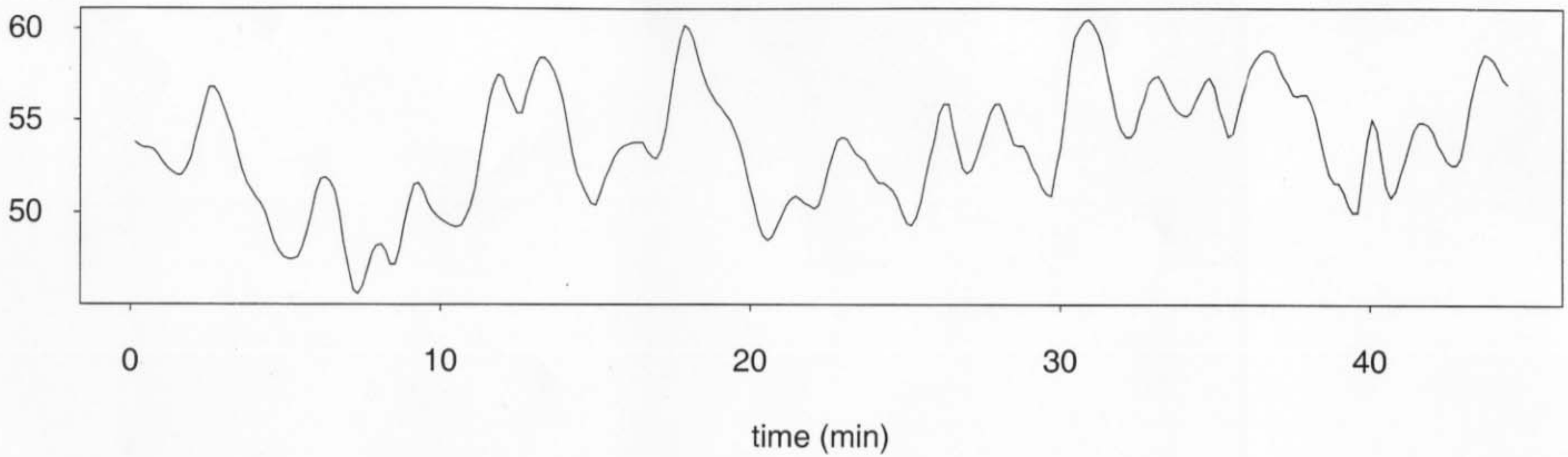
Monthly mean sunspot numbers, 1861-1960



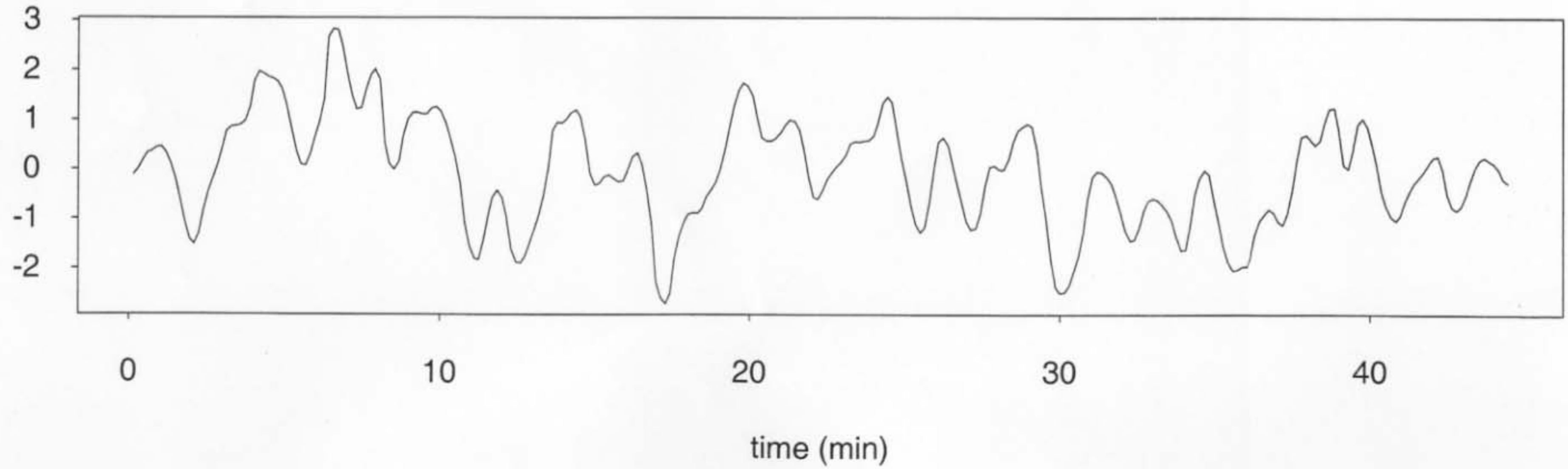

Coherence
coherence
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
frequency

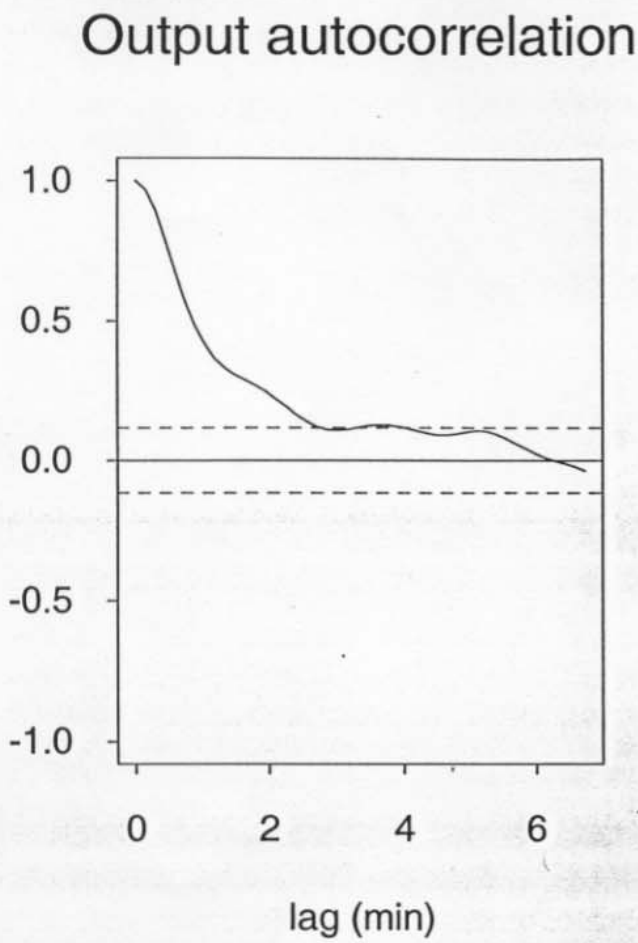

Sunspot autocorrelation
Mississippi autocorrelation
1.0
0.5
0.0
-0.5
-1.0
0 20 40 60 80 120
lag
1.0
0.5
0.0
-0.5
-1.0
0 20 40 60 80 120
lag
Crosscorrelation
1.0
0.5
0.0
-0.5
-1.0
-100 -50 0 50 100
lag

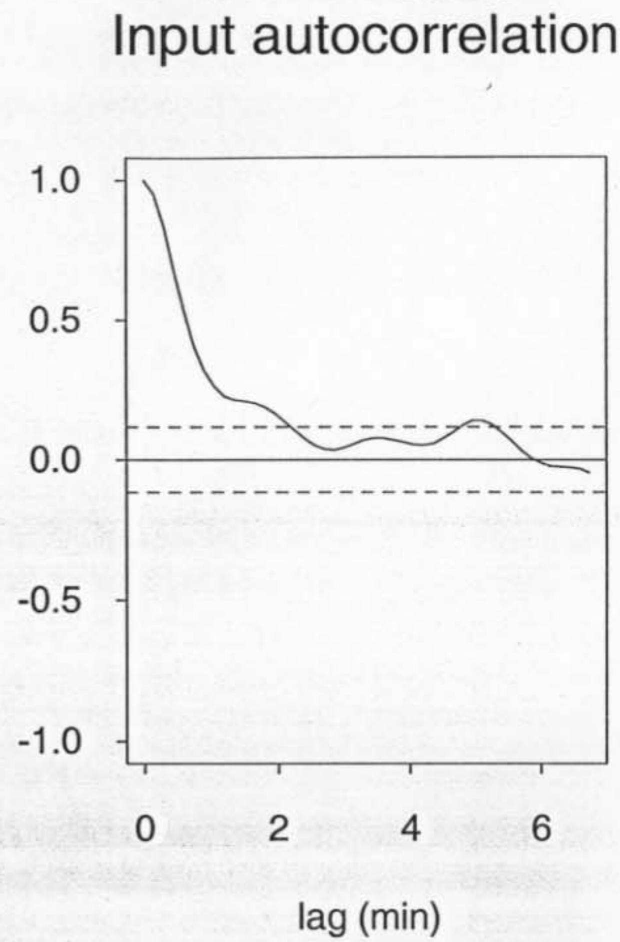
Percent CO2 in outlet gas

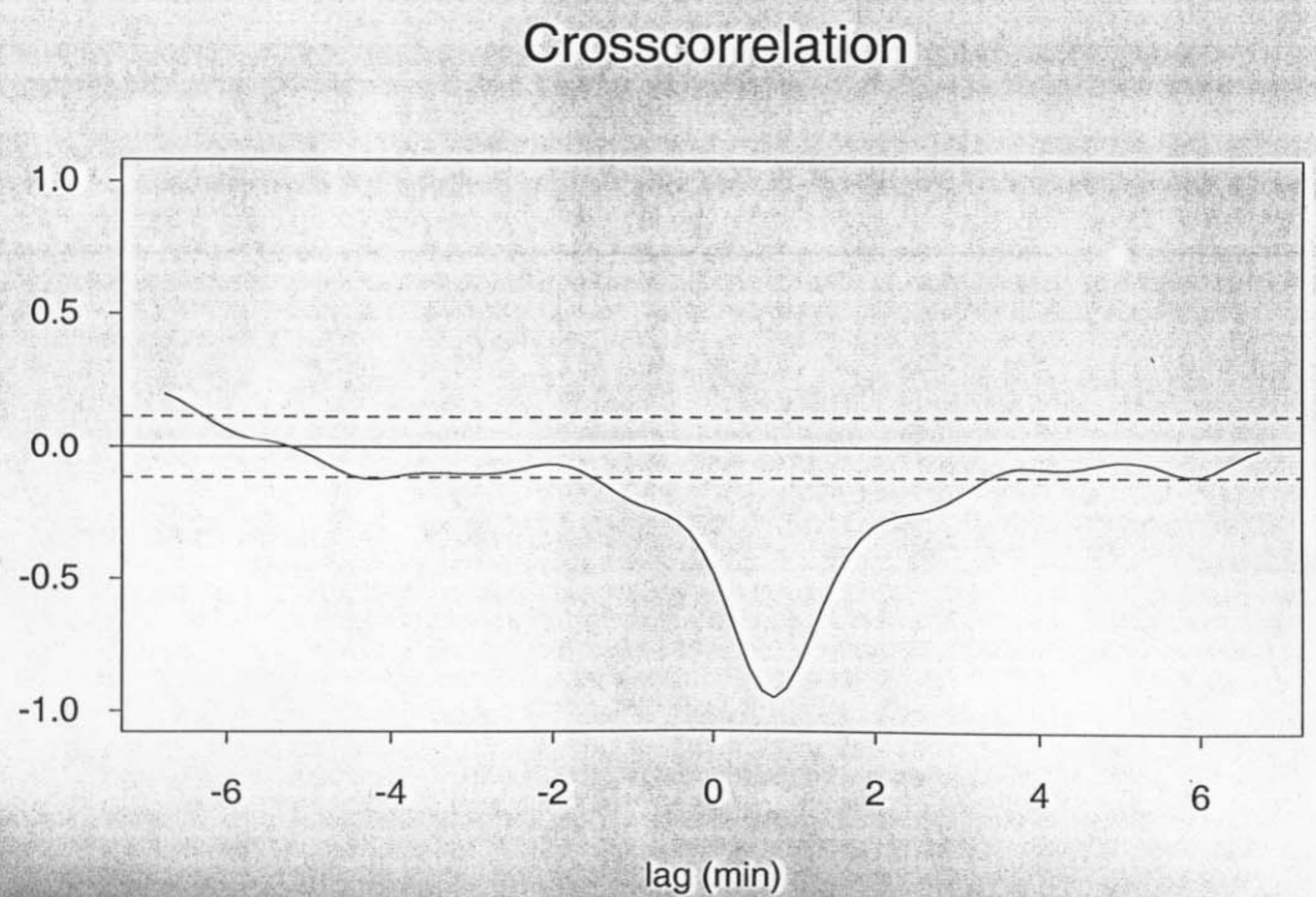


(.6 - methane feed)/.04

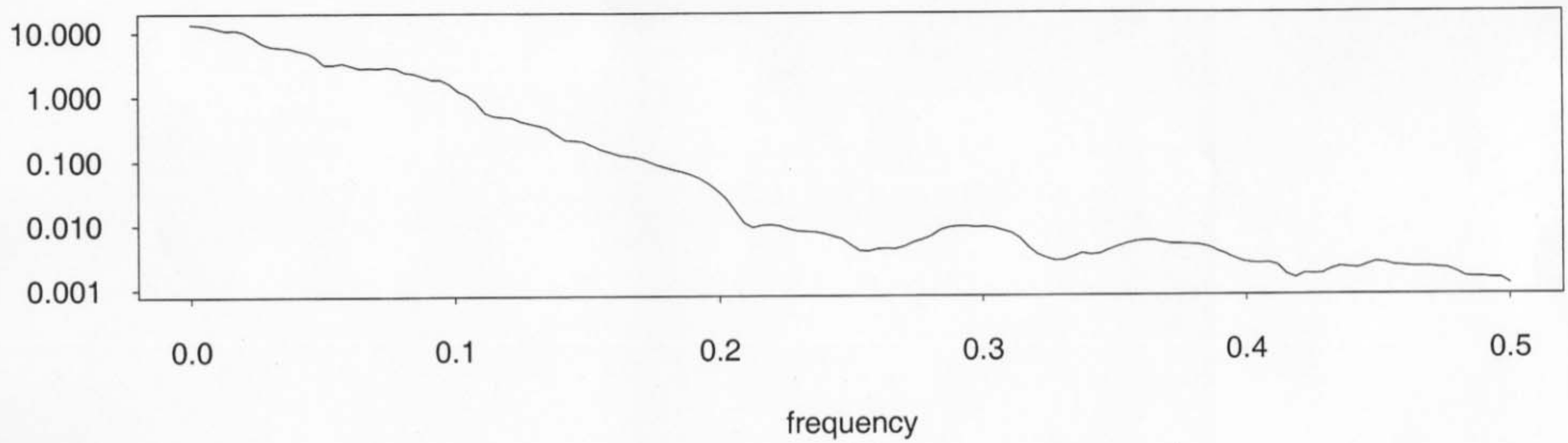



Output autocorrelation
1.0
0.5
0.0
-0.5
-1.0
0
2
4
6
lag (min)

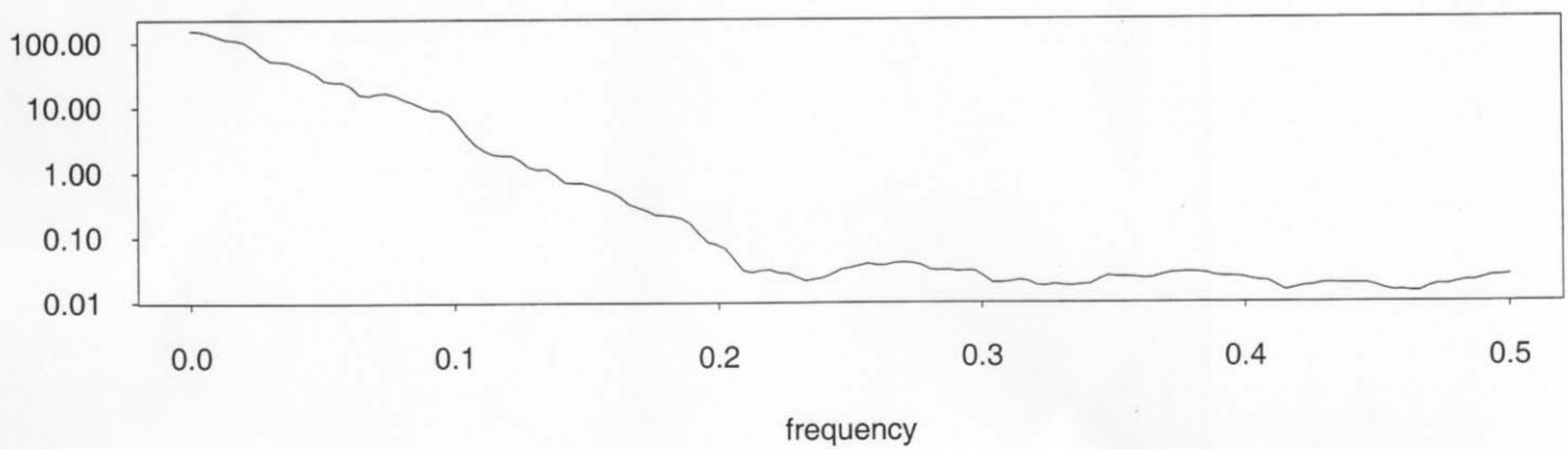

Input autocorrelation
1.0
0.5
0.0
-0.5
-1.0
0
2
4
6
lag (min)

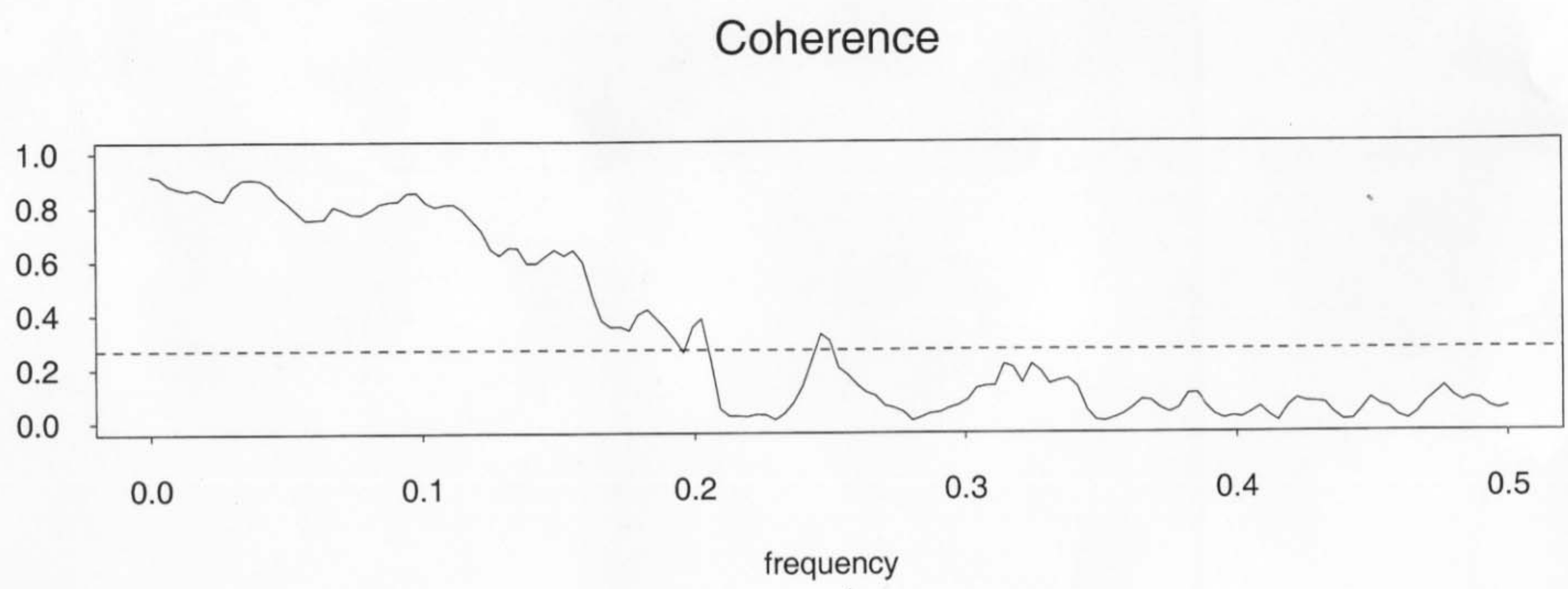

Crosscorrelation
1.0
0.5
0.0
-0.5
-1.0
-6
-4
-2
0
2
4
6
lag (min)

Input spectrum

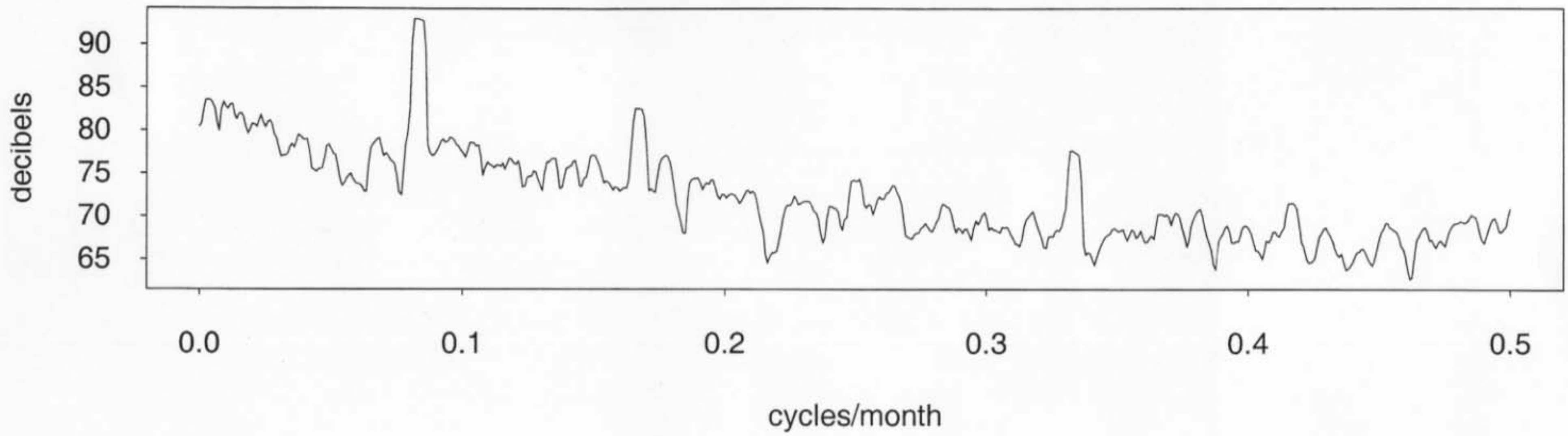


Output spectrum




Coherence
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
frequency

Runoff spectrum



Sunspot spectrum