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Third-order spectra

$(X(t), Y(t), Z(t))$ $t = 0, \pm 1, \pm 2, \ldots$
stationary
zero mean

third order cumulant

$$E\{X(t+u)Y(t+v)Z(t)\} = c_{XYZ}(u,v)$$

Cramér representation

$$X(t) = \int_{-\pi}^{\pi} e^{it\lambda}\, dZ_X(\lambda)$$

$$E\{dZ_X(\lambda)\, dZ_Y(\mu)\, dZ_Z(\nu)\}$$

$$= \delta(\lambda+\mu+\nu) f_{XYZ}(\lambda,\mu)\, d\lambda\, d\mu\, d\nu$$

(λ,μ) bifrequency

Uses: system identification
checking for linearity
checking for gaussianity
fitting parametric models

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Genesis of bifrequencies

$$X(t) = \mu + \alpha \cos(\lambda t + \beta) + \gamma \cos(\mu t + \delta)$$

$$Y(t) = X(t)^2 = \mu^2 + 2\mu\alpha\cos(\lambda t+\beta) + 2\mu\gamma\cos(\mu t+\delta)$$

$$+ \alpha^2 \cos(\lambda t+\beta)^2 + 2\alpha\gamma\cos(\lambda t+\beta)\cos(\mu t+\delta)$$

$$+ \gamma^2 \cos(\mu t+\delta)^2 \qquad (*)$$

Identities.

$$\cos x^2 = (1+\cos 2x)/2$$

$$\cos x \cos y = (\cos(x+y) + \cos(x-y))/2$$

$(*)$ involves frequencies:

$\lambda, \mu, 2\lambda, \lambda+\mu, \lambda-\mu, 2\mu$

and their negatives

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(I)

Estimation of the biospectrum.

Data

$Y(t),\ t = 0, 1, \ldots, T-1$

$T = LV$

L stretches of length V

$$d^V(\lambda; l) = \sum_{v=0}^{V-1} h\left(\frac{v+1}{V+1}\right) Y(lV+v) e^{-iv\lambda}$$

h: 0 outside 0,1

Periodogram of order 3

$$I^V_{YYY}(\lambda, \mu; l) = \frac{1}{(2\pi)^3 V h_3} d^V_Y(\lambda; l) d^V_Y(\mu; l) \overline{d^V_Y(\lambda+\mu; l)}$$

$$f^T_{YYY}(\lambda, \mu) = \frac{1}{L} \sum_{l=0}^{L-1} I_{YYY}(\lambda, \mu; l)$$

$$\sim N^c\left(f_{YYY}(\lambda, \mu),\ \frac{h_6}{h_3^2} f_{YY}(\mu) f_{YY}(\lambda) f_{YY}(\lambda+\mu) \frac{V}{L}\right)$$

$V,\ L/V \to \infty$ as $T \to \infty$

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When $f_{YYY}(\lambda,\mu) \equiv 0$

$$|f^T_{YYY}(\lambda,\mu)|^2 \sim \frac{h_6}{h_3^2} \frac{V}{L} \frac{1}{2\pi} f_{YY}(\lambda) f_{YY}(\mu) f_{YY}(\lambda+\mu) \frac{\chi_2^2}{2}$$

Gaussianity

$$f_{YYY}(\lambda, \mu) \equiv 0$$

$$|f_{YYY}(\lambda, \mu)|^2 \equiv 0$$

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Linearity

$$Y(t) = \sum_u a(u)\, X(t-u), \quad X : WN$$

$$f_{YY}(\lambda) = |A(\lambda)|^2 \frac{\sigma^2}{2\pi}$$

$$f_{YYY}(\lambda,\mu) = A(\lambda)A(\mu)\overline{A(\lambda+\mu)}\frac{K_3}{(2\pi)^2}$$

Bicoherence

$$\frac{|f_{YYY}(\lambda,\mu)|^2}{f_{YY}(\lambda)f_{YY}(\mu)f_{YY}(\lambda+\mu)} = \frac{K_3^2}{\sigma^6} \cdot \frac{(2\pi)^3}{(2\pi)^4}$$

i.e. constant

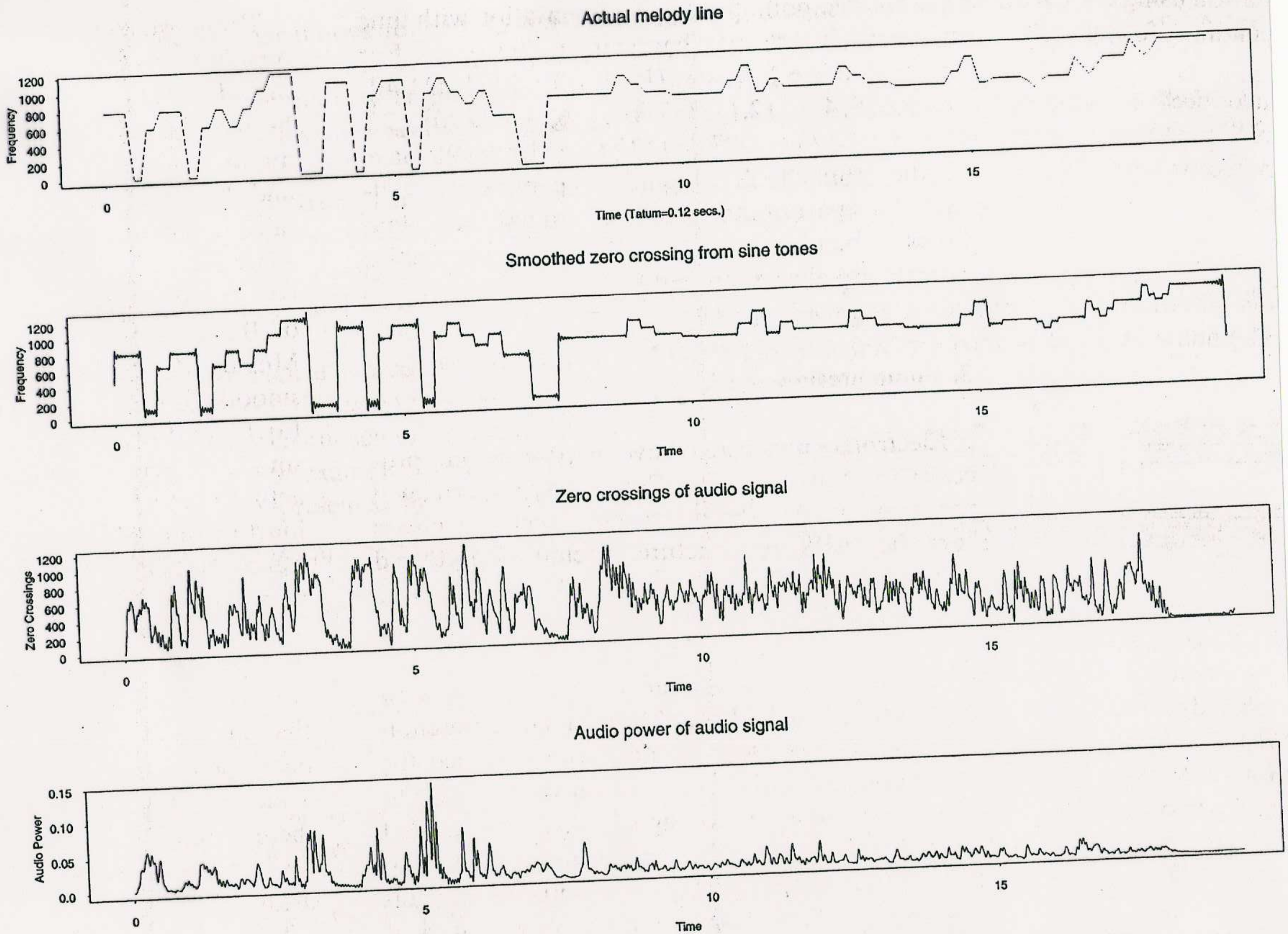

Fig. 1. Score, smoothed zero crossings and instantaneous audio power of Eine Kleine Nachtmusik.

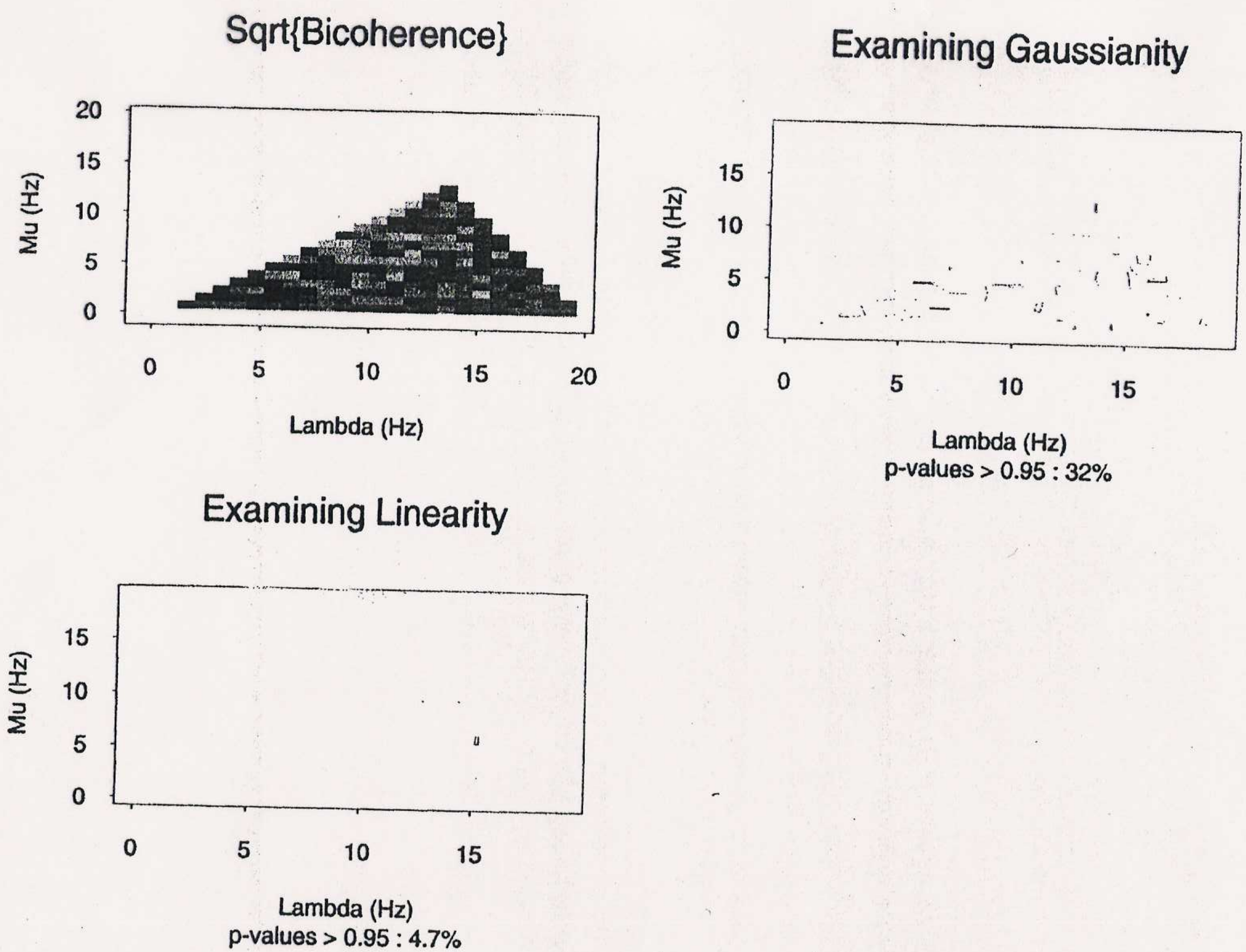

Fig. 7. Square root of the bicoherence estimate and contour plots of prob-values for non-Gaussianity and nonlinearity, respectively, for the smoothed zero crossings of the *Coffee Cantata.*

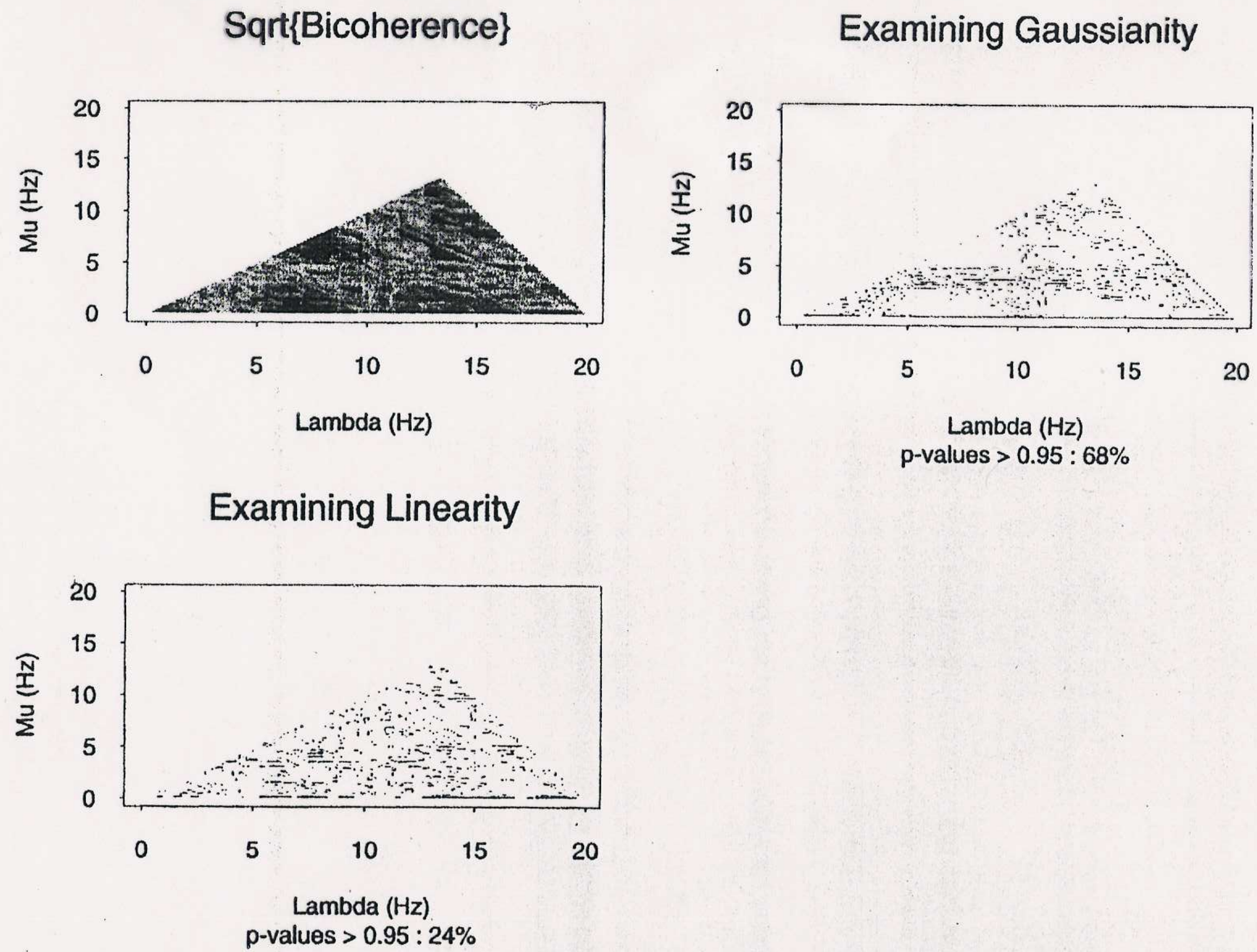

Fig. 8. Square root of the bicoherence estimates and contour plots of prob-values for non-Gaussianity and nonlinearity, respectively, fo the instantaneous power obtained from the *Coffee Cantata* signal.

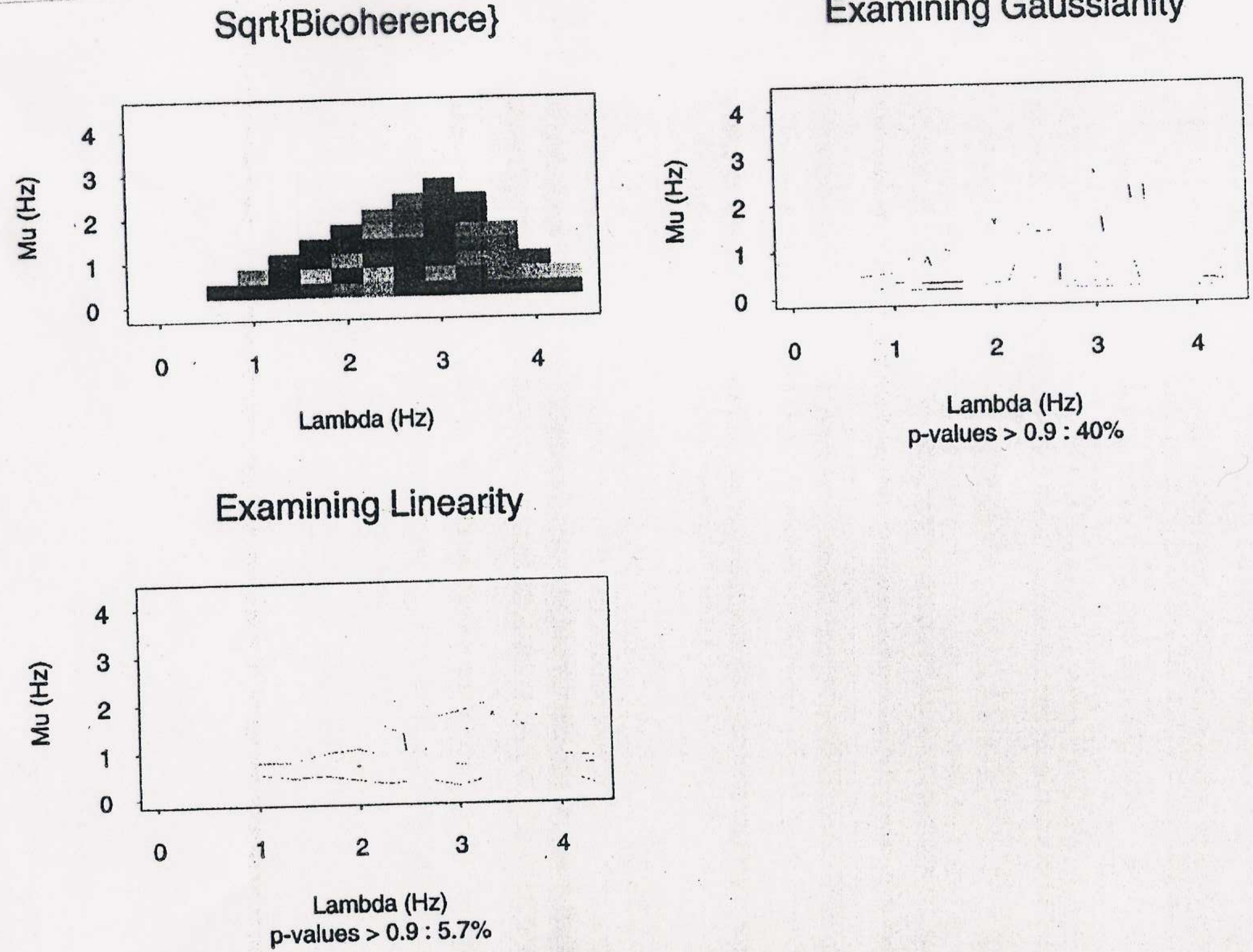

Fig. 9. Square root of the bicoherence estimates and contour plots of prob-values for non-Gaussianity and nonlinearity for the score of the *Coffee Cantata*.

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<u>Bilinear system.</u>

$$\frac{d\underset{\sim}{S}(t)}{dt} = \underset{\sim}{a} + \underset{\sim}{A}\,\underset{\sim}{S}(t) + \underset{\sim}{B}\,\underset{\sim}{S}(t)\,X(t)$$

$$Y(t) = \mathrm{Re}\{\underset{\sim}{c}^T \underset{\sim}{S}(t)\}$$

$\underset{\sim}{S}, \underset{\sim}{a}$ .. complex-valued entries

$$\underset{\sim}{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

$$A_j = \beta_j + i\gamma_j$$

$$\gamma_1, \gamma_2 = \frac{2\pi}{15}, \frac{2\pi}{5}$$

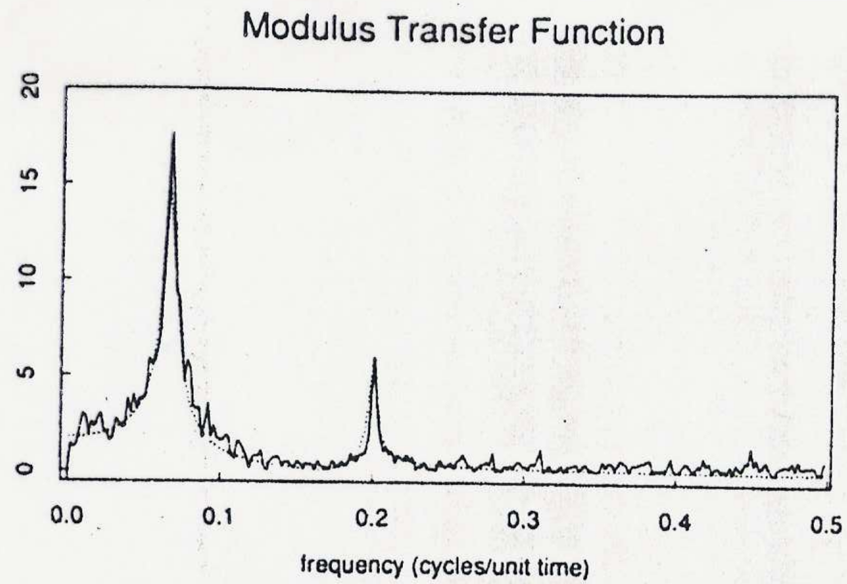
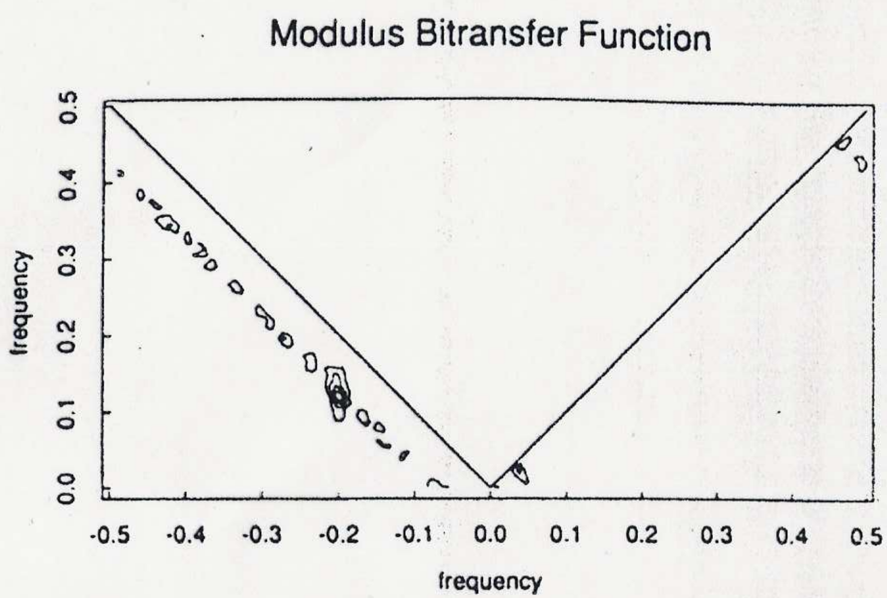


Fig. 6. The function $|H_1(\lambda)|$ as computed from (23), folded for aliasing, using the maximum likelihood estimates of the parameters. The curve is superposed on the bottom display of Fig. 2.



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System identification.

$$Y(t) = \mu + \int a(u) X(t-u) du$$

$$+ \iint b(u,v) X(t-u) X(t-v) du dv + \epsilon(t)$$

(X stationary Gaussian

$\perp\!\!\!\perp$ stationary ϵ

$$f_{YX}(\lambda) = A(\lambda) f_{XX}(\lambda)$$

$$f_{XXY}(-\lambda, -\mu) = 2 B(-\lambda, -\mu) f_{XX}(\lambda) f_{XX}(\mu)$$

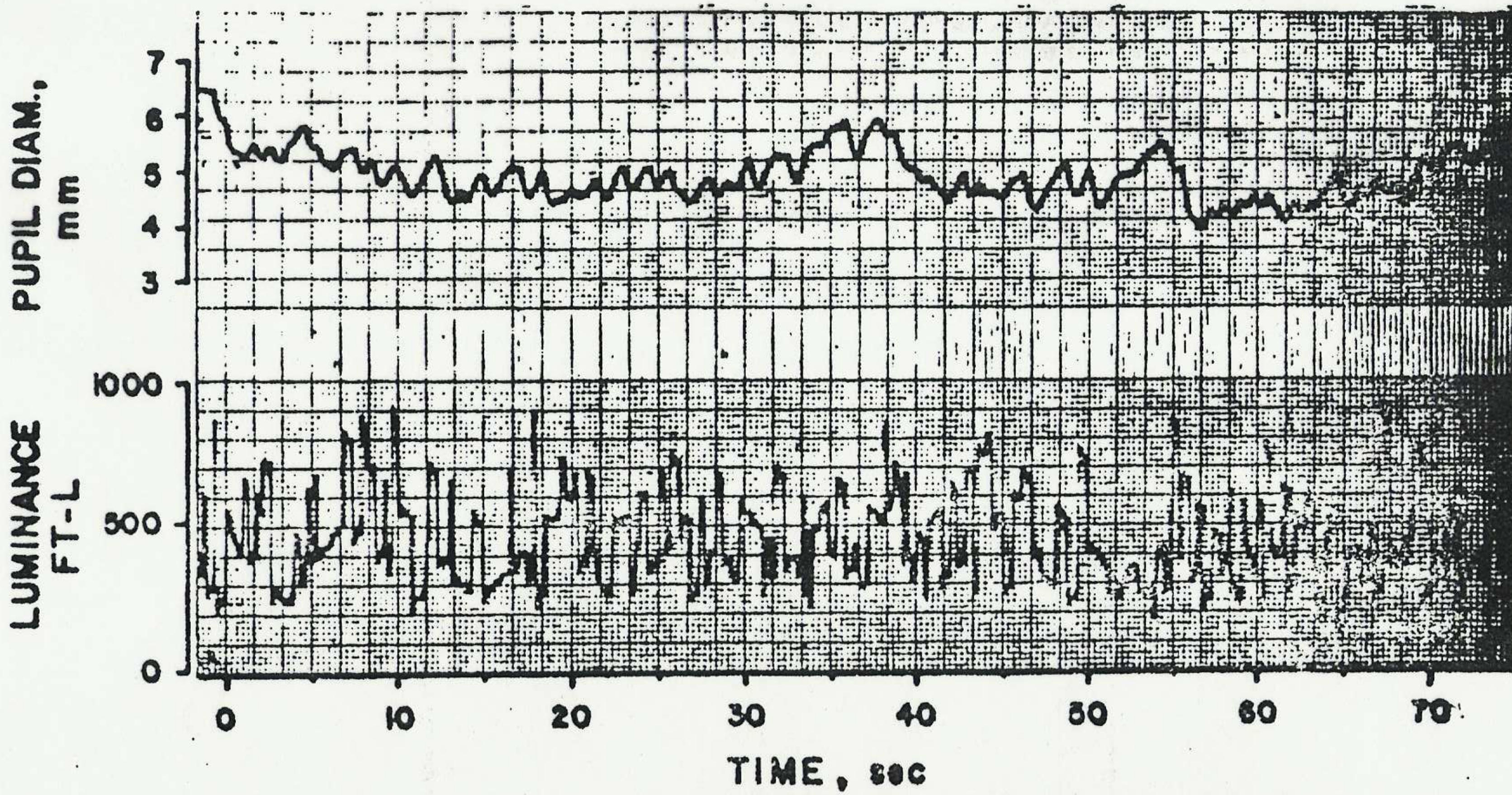

FIG. 6. Raw data showing rapidly changing stimulus low-pass filtered at 5.0 Hz with slope 24 dB/octave and associated pupillary response. Subject BH.

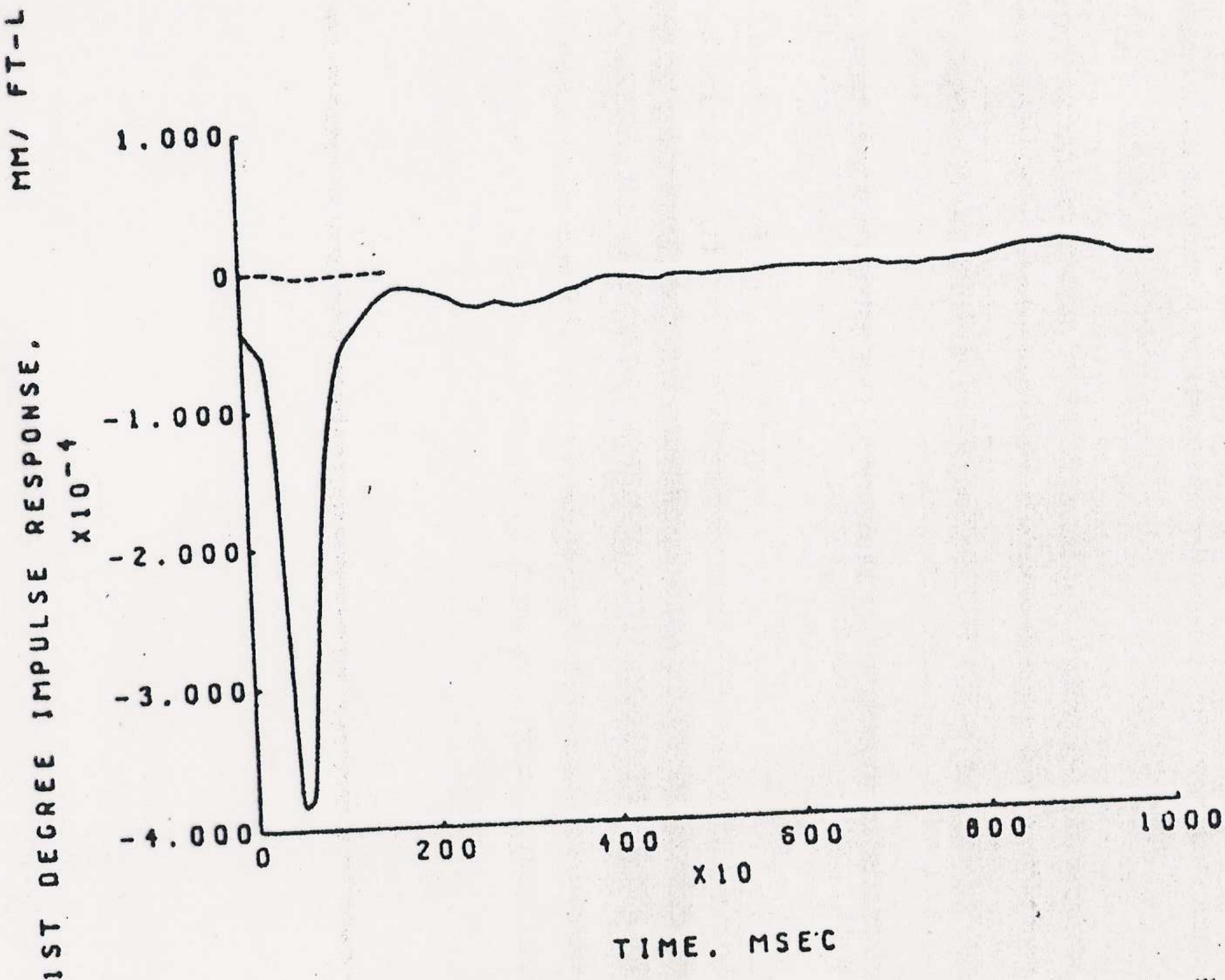

FIG. 8. 1st-degree kernels (solid line) and 2nd-degree main-diagonal kernels (dashed line) of the pupillary system obtained from spectral program analysis of random-stimulus pupil experimental data. Subject BH.

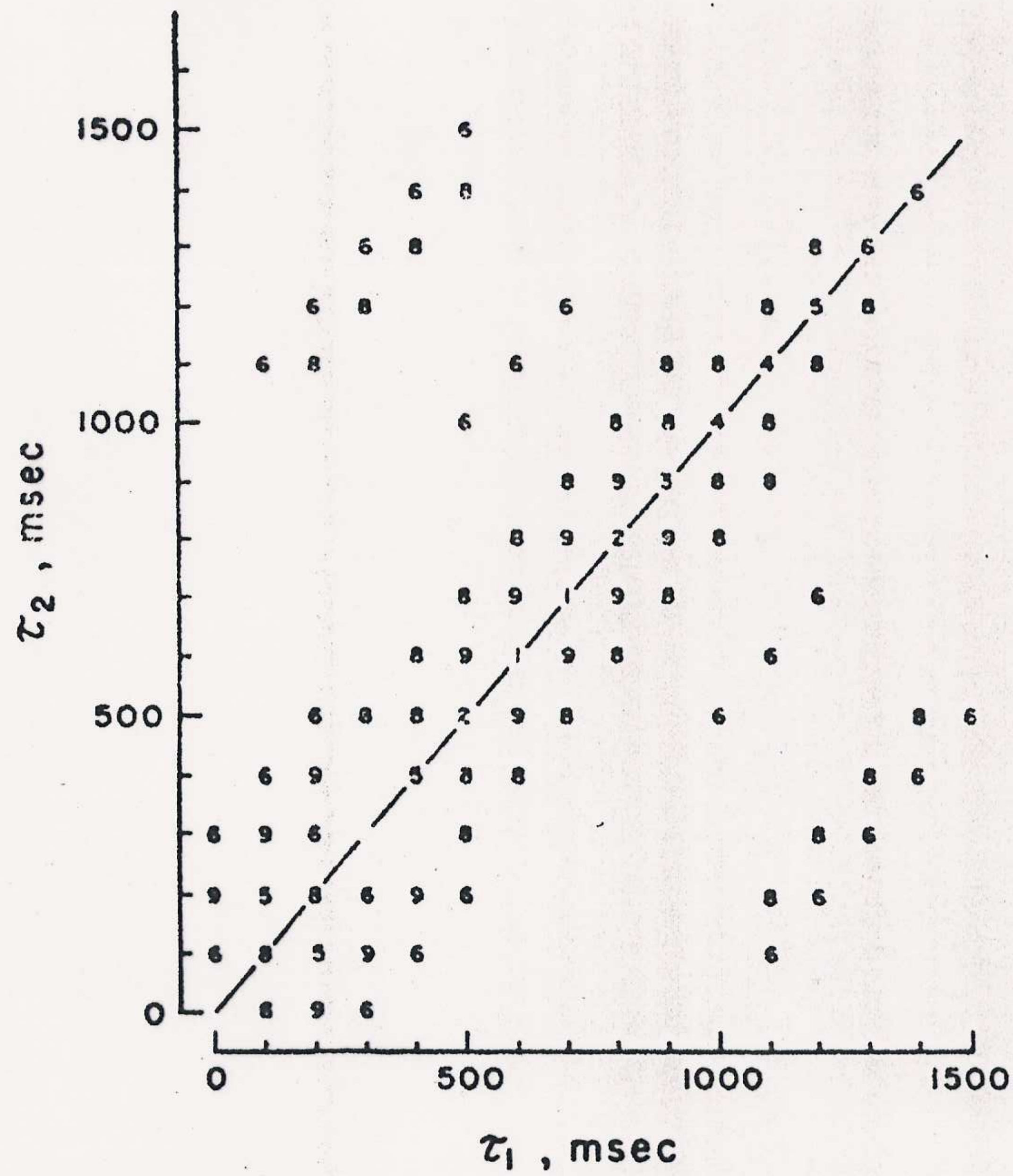

FIG. 9. Contour print of pupil 2nd-degree kernels (spectral calculation method). The dashed line indicates the main diagonal. The number symbol for the most common level (7) is not plotted. Numerical values are as follows:

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Quadratic coherence

$$\frac{1}{2 f_{YY}(\lambda)} \int_0^{\Lambda} \frac{| f_{XXY}(\lambda-\omega, \omega)|^2}{f_{XX}(\lambda-\omega) f_{XX}(\omega)} d\omega$$

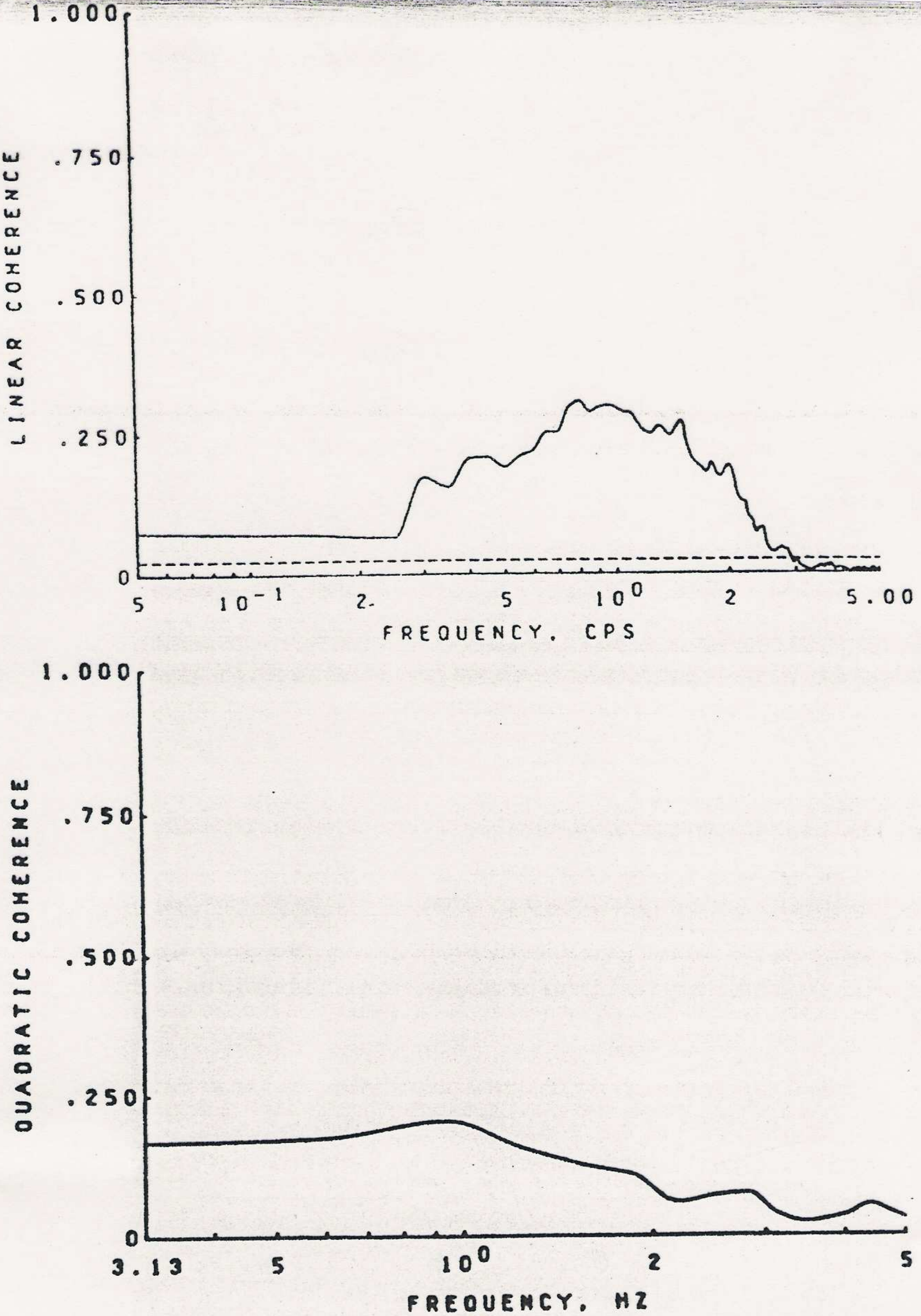

FIG. 7. (a) 1st-degree (linear) coherence, indicating amount of linear time invariant correspondence between input and output as a function of frequency. Subject BH. The horizontal dashed line at 0.0245 gives the 95% null point. (b) 2nd-degree (quadratic) coherence, indicating amount of quadratic effect on the output due to the input as a function of frequency. Note that peaks occur at about 0.9 Hz for both linear and quadratic coherence. Subject BH.

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Parameter estimation

$$I_s^T = I_{YY}^T\left(\frac{2\pi s}{T}\right)$$

$$I_{r,s}^T = I_{YYY}^T\left(\frac{2\pi r}{T}, \frac{2\pi s}{T}\right)$$

cp. Whittle estimation*

Solve

$$\sum_s \frac{1}{f_s^2}(I_s^T - f_s)\frac{\partial f_s}{\partial \theta}$$

$$+ \frac{2\pi}{T}\sum_r \sum_s \frac{1}{f_r f_s f_{r+s}}(I_{r,s}^T - f_{r,s})\frac{\partial f_{r,s}}{\partial \theta} = 0$$

Residuals $\frac{2\pi}{T}|I_{r,s}^T - \hat{f}_{r,s}|^2 / \hat{f}_r \hat{f}_s \hat{f}_{r+s}$

$$\hat{f}_{r,s} = f_{YYY}\left(\frac{2\pi r}{T}, \frac{2\pi s}{T}; \hat{\theta}\right)$$

* $\min \sum_s (\log f_s + I_s^T / f_s)$

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Luteinizing hormone   Cow

$\Delta t = 10$ min

$$Y(t) = \sum_j a(t - \sigma_j)$$

$\{\sigma_j\}$ unobserved , $a(t) = e^{-\alpha t}$

$$f_2(\lambda) = \frac{\sigma^2}{2\pi} \frac{1 - |\zeta|^2}{|1 - \zeta|^2} \frac{1}{2\alpha}$$

$$f_3(\lambda, \mu) = \frac{\gamma \sigma^3}{(2\pi)^2} \frac{1 - \zeta_1 \zeta_2 \zeta_3}{(1 - \zeta_1)(1 - \zeta_2)(1 - \zeta_3)} \frac{1}{3\alpha}$$

$$\zeta = \exp\{-(\alpha + i\lambda)\} = \zeta_1$$

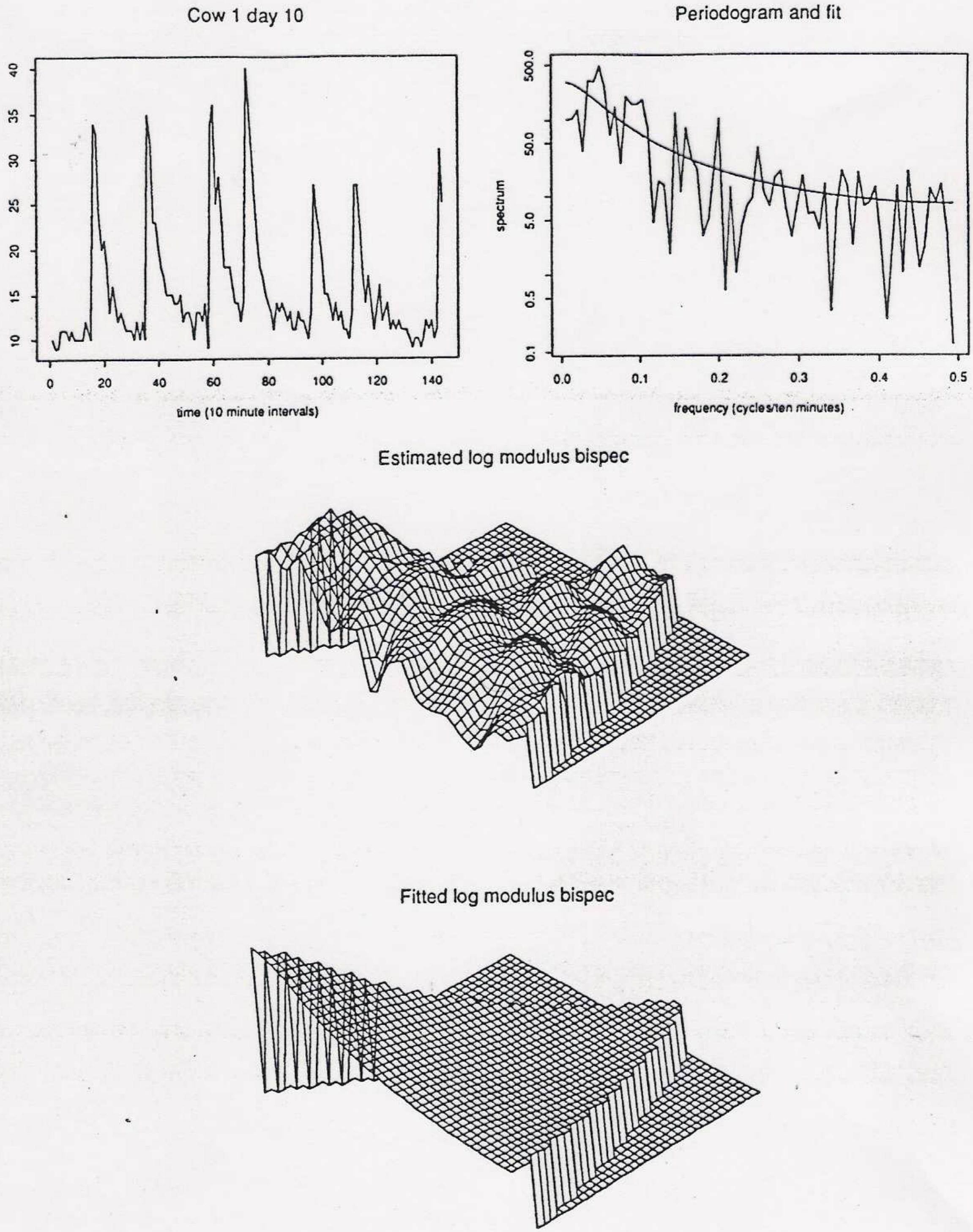
$$\zeta_2 = \exp\{-(\alpha + i\mu)\} \quad \zeta_3 = \exp\{-(\alpha - i\{\lambda + \mu\})\}$$

$$\text{cov}\{dM(t+u), dM(t)\} = \sigma^2 \delta(u)\, dt\, du$$

$$\text{cum}\{dM(t+u), dM(t+v), dM(t)\}$$

$$= \gamma \sigma^3 \delta(u) \delta(v)\, dt\, du\, dv$$

Figure 2


Cow 1 day 10
time (10 minute intervals)
Periodogram and fit
spectrum
frequency (cycles/ten minutes)
Estimated log modulus bispec
Fitted log modulus bispec

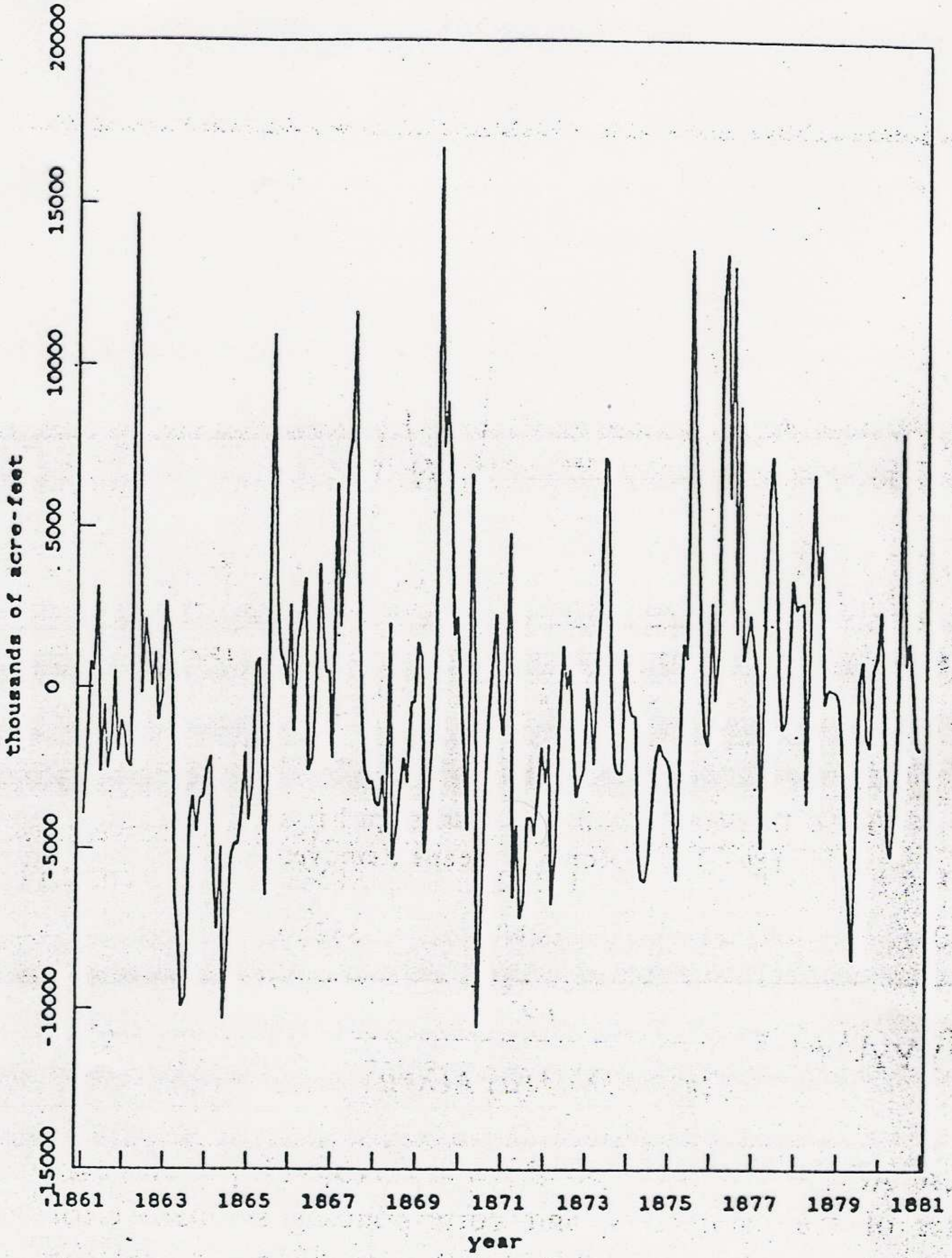


Figure 2. Mississippi River Runoff, 1861-1980,
Monthly Means Removed.

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Mississippi River

seasonally adjusted

$\Delta t = 1$ month

AR(2)

$$f_{YY}(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|1 + \alpha_1 e^{-i\lambda} + \alpha_2 e^{-i2\lambda}|^2}$$

$$f_{YYY}(\lambda) = \frac{\gamma\sigma^3}{(2\pi)^2} \frac{1}{A(\lambda) A(\mu) \overline{A(\lambda+\mu)}}$$

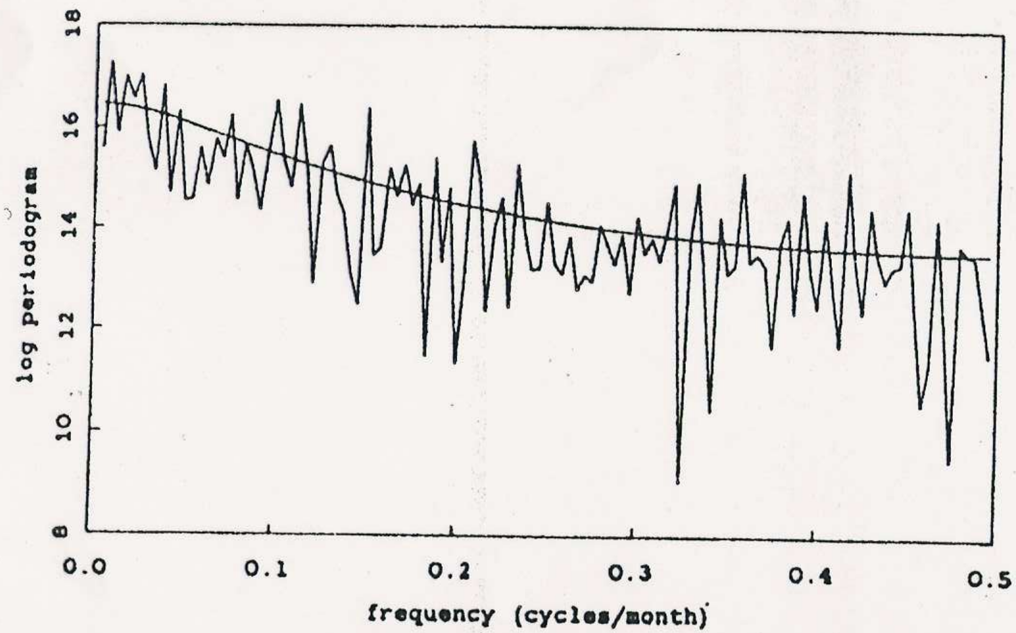

Figure 6. Log Periodogram and Fitted Autoregressive of Order 2, Data of Figure 2.

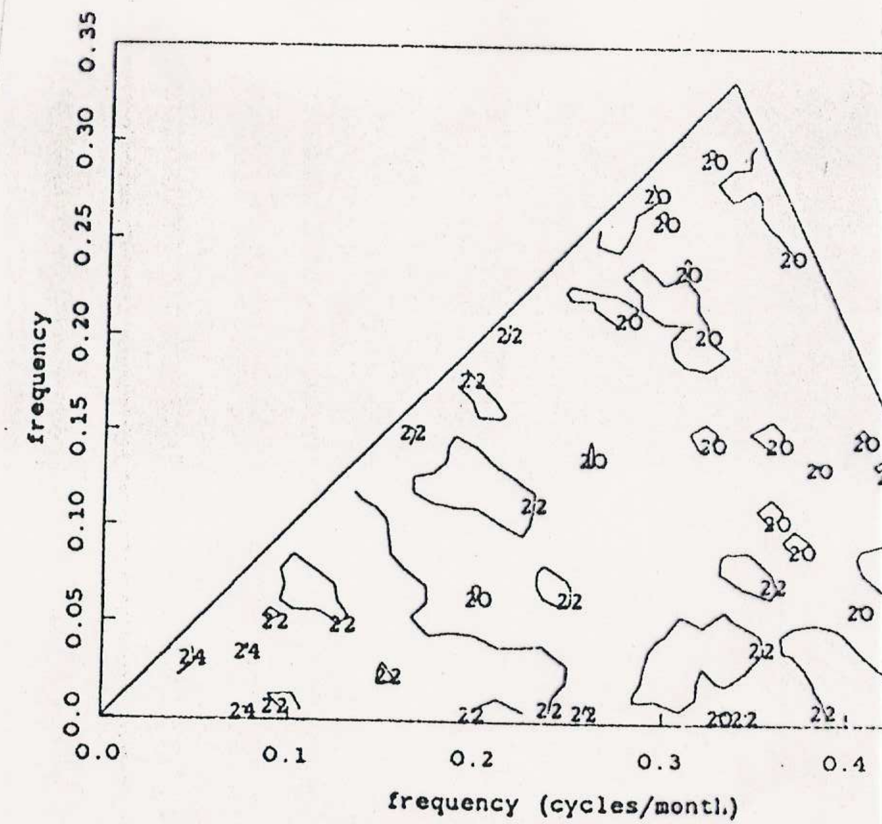

Figure 8. Log Modulus of Estimated Bispectrum, Data of Figure 2.

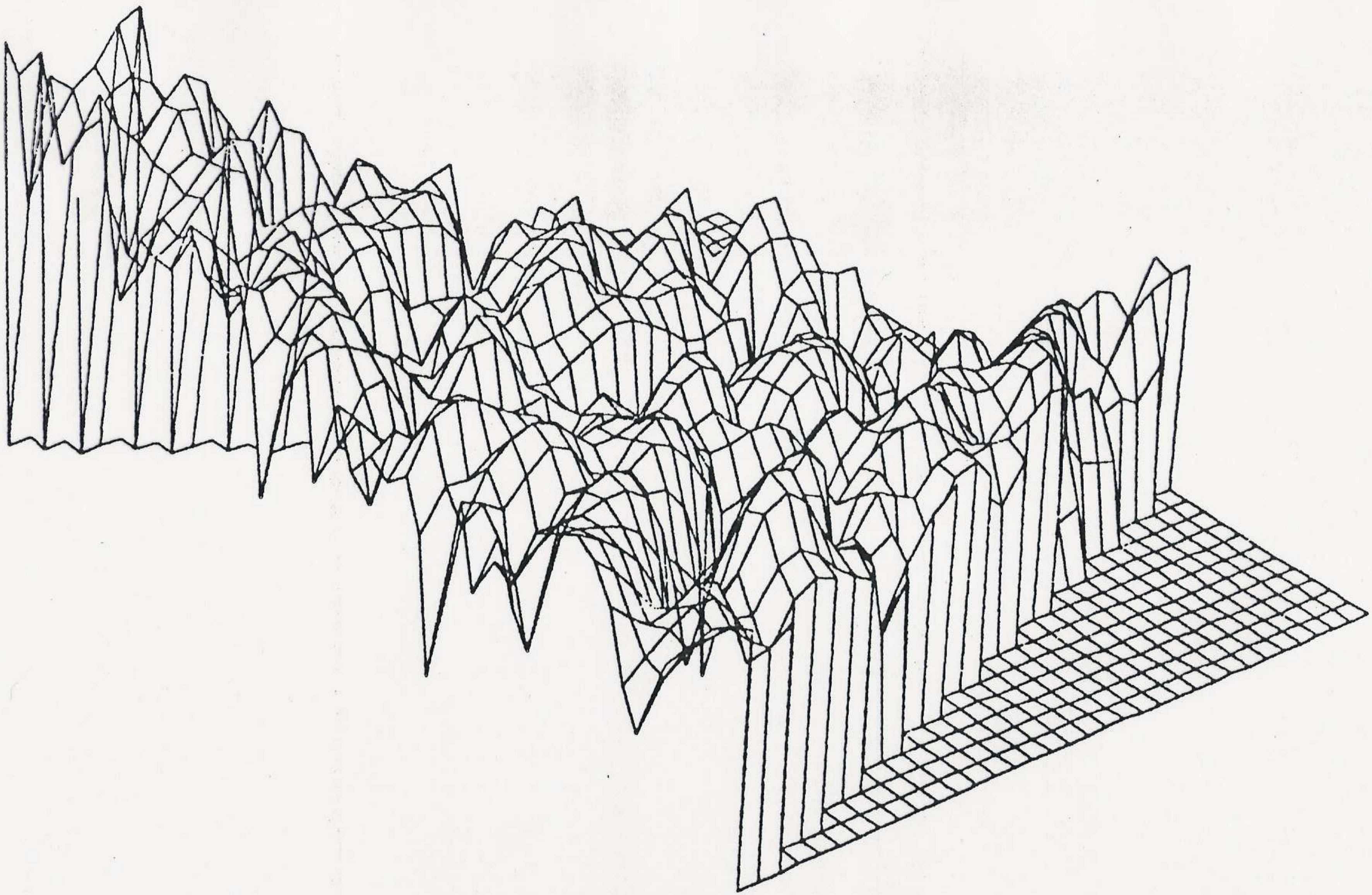

Figure 10. Perspective Plot Corresponding to Figure 8.

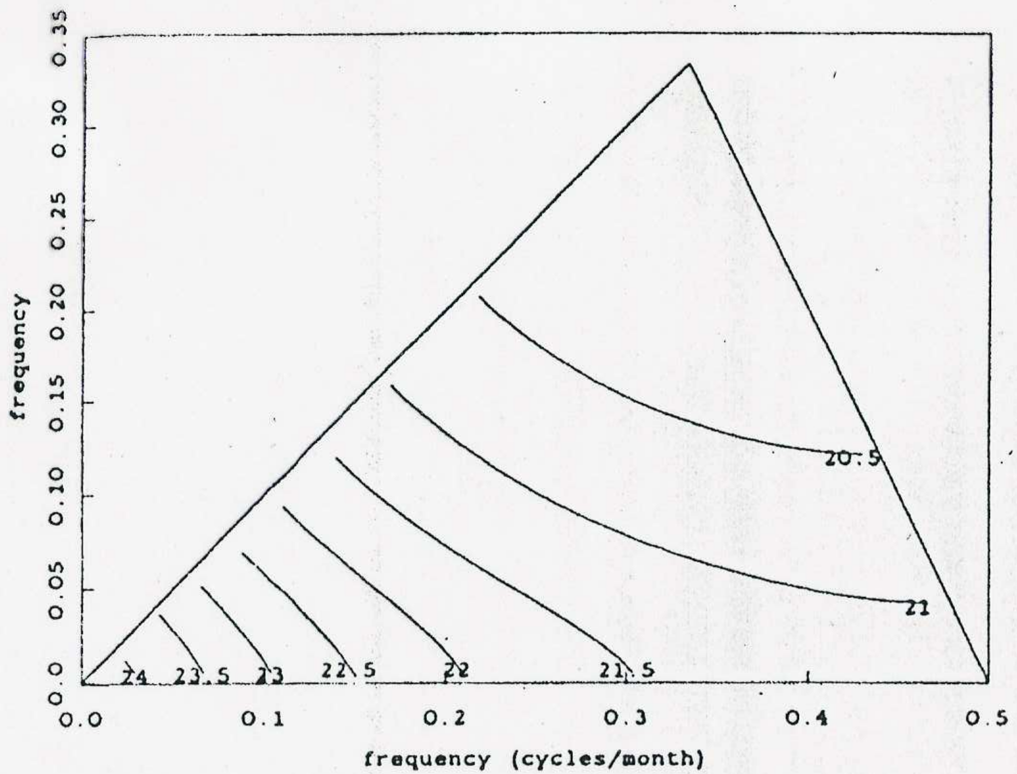

Figure 9. Log Modulus of Fitted Bispectrum.

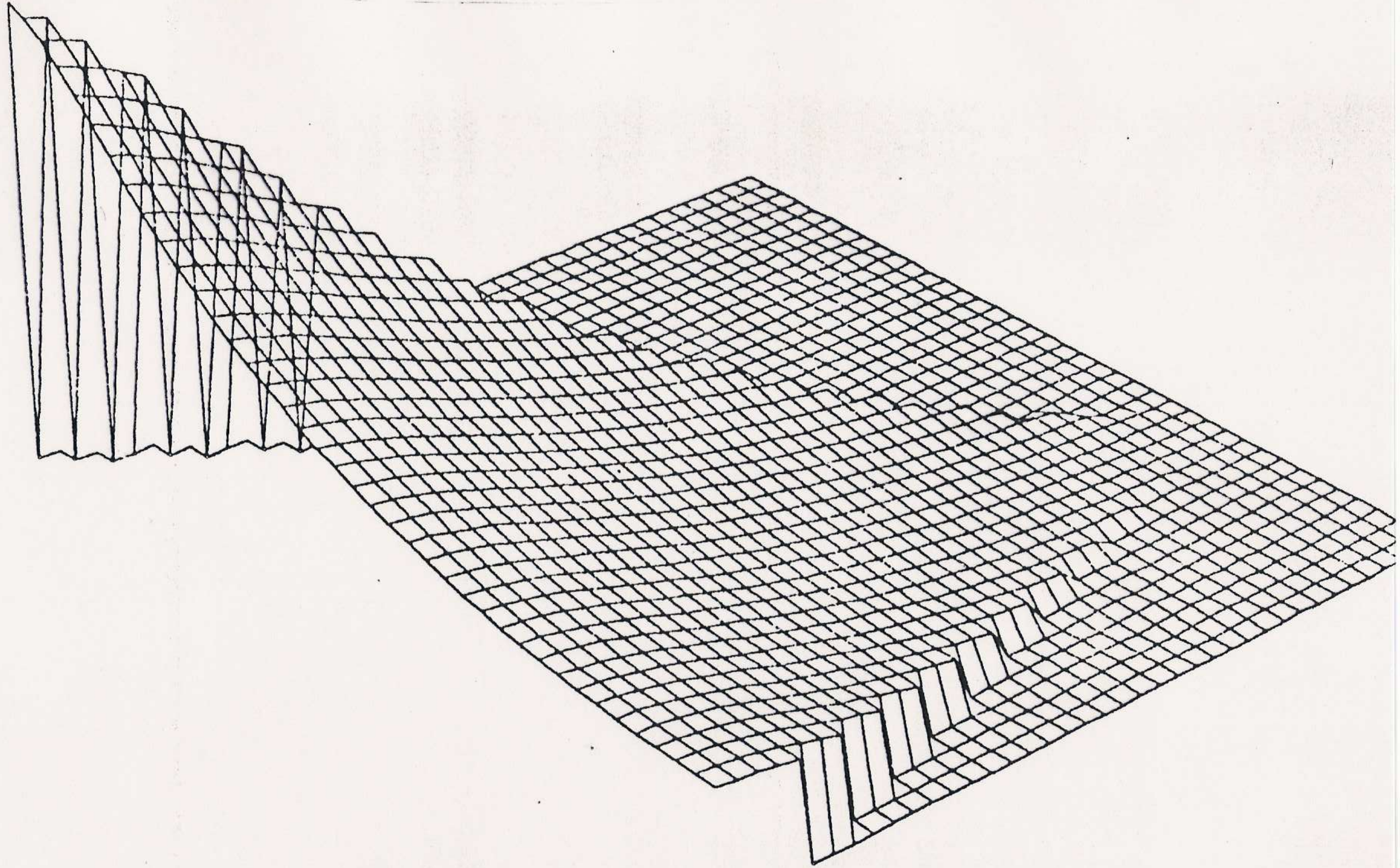

Figure 11. Perspective Plot Corresponding to Figure 9.

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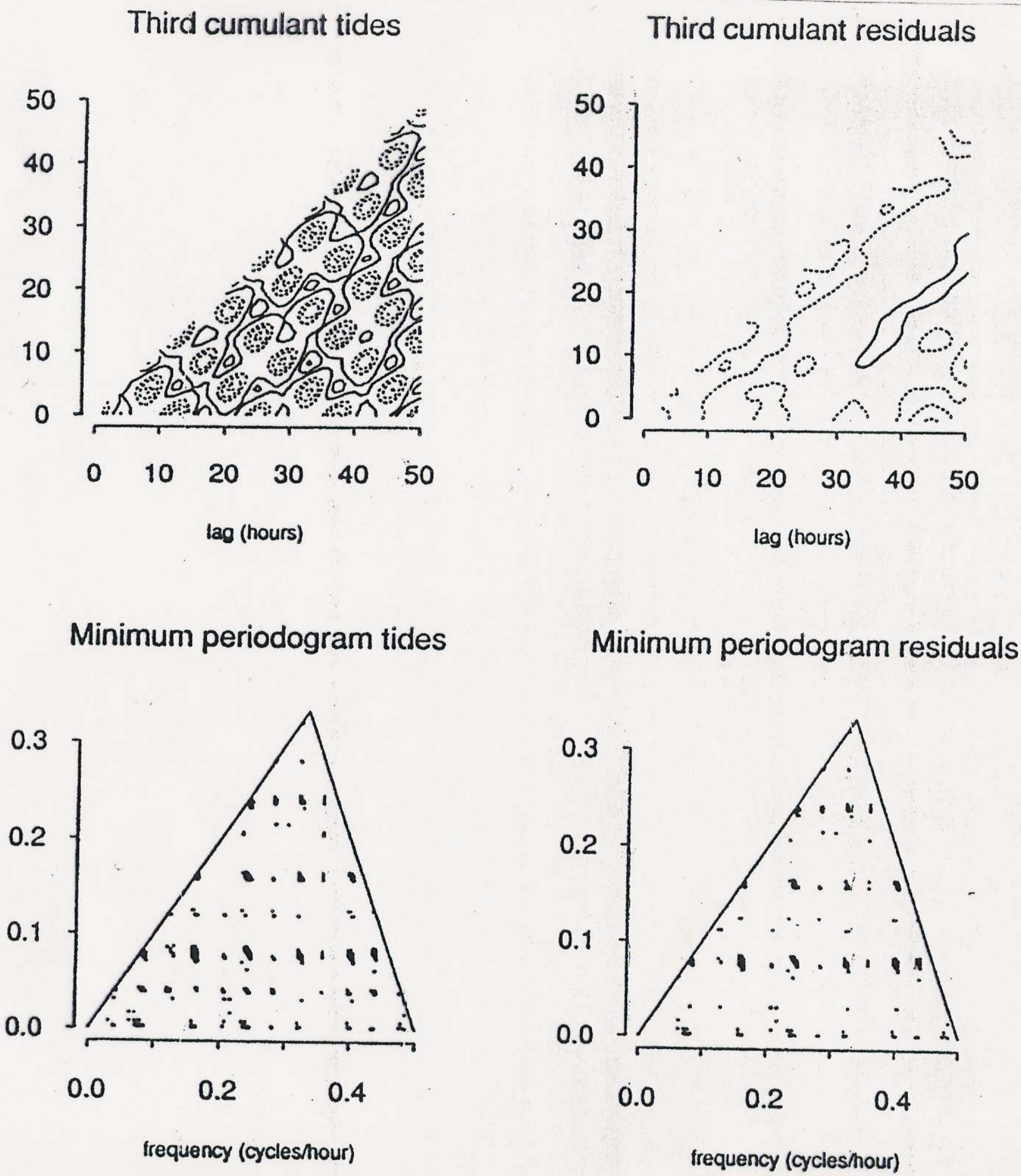
Variants. Minimum periodogram

$$\min\left\{ \frac{I^T(\lambda)}{f^T(\lambda)}, \frac{I^T(\mu)}{f^T(\mu)}, \frac{I^T(\lambda+\mu)}{f^T(\lambda+\mu)} \right\}$$

f^T: heavily smoothed

H_0: at least one of $\lambda, \mu, \lambda+\mu$ absent


Third cumulant tides
50
40
30
20
10
0
0 10 20 30 40 50
lag (hours)
Third cumulant residuals
50
40
30
20
10
0
0 10 20 30 40 50
lag (hours)
Minimum periodogram tides
0.3
0.2
0.1
0.0
0.0 0.2 0.4
frequency (cycles/hour)
Minimum periodogram residuals
0.3
0.2
0.1
0.0
0.0 0.2 0.4
frequency (cycles/hour)