
[Jumping ahead]

[† The R package named tseries is required for this chapter. We assume that the reader has downloaded and installed it.]

The models discussed so far concern the conditional mean structure of time series data.

However, more recently, there has been much work on modeling the conditional variance structure of time series data.

In finance, the conditional variance/conditional volatility of the return, \( r_t \), of a financial asset is often adopted as a measure of the risk of the asset

A key component in the mathematical theory of pricing a financial asset and the VaR (Value at Risk) calculations, Tsay book

In an “efficient market”, the expected return (conditional mean) should be zero, and hence the return series should be white noise.

[Efficient market. all pertinent information is available to all participants at the same time, and prices respond immediately to available information]
12.1 Some Common Features of Financial Time Series

Consider the daily values of a unit of the CREF stock fund over the period from August 26, 2004 to August 15, 2006.

The CREF stock fund is a fund of several thousand stocks and is not openly traded in the stock market.

Since stocks are not traded over weekends or on holidays, only on so-called trading days, the CREF data do not change over weekends and holidays.

No weekend effect

Return. \( p_t \) price

\[
    r_t = \log(p_t) - \log(p_{t-1}) \\
    \approx \frac{\text{change}(p_t)}{p_{t-1}}
\]
Volatile. Series liable to change rapidly and unpredictably.

Returns more volatile over some time periods and above became very volatile toward end of the study period.

On their website Merrill Lynch provides the following definition,

“Volatility. A measure of the fluctuation in the market price of the underlying security. Mathematically, volatility is the annualized standard deviation of returns.”

Pattern of alternating quiet and volatile periods of substantial duration is referred to as volatility clustering.
The study of volatility can lead to better forecasting of a series, to better understanding of the past, and to better assessment of risk. For example in insurance considerations the safety loaded pure risk premium can take the form

$$\lambda_1 p(t) + \lambda_2 \sigma(t) + \lambda_3 \sigma(t)^2$$

where the $\lambda$’s are weights, $p(t)$ is the fair premium, and $\sigma(t)$ and $\sigma(t)^2$, are volatility measures at time $t$. One reference to the insurance case is Daykin et al (1994).

Volatility may also be formalized as the conditional variance

$$\sigma^2_{t|t-1}$$

It varies over time

A medical example. Atrial fibrilations

![R-R intervals graph](image)

Figure 1: 6000 RR interval lengths plotted against beat number.
An empirical formula for volatility at time $t$ is provided by $	ext{se}\{Y_s \mid s \text { near } t\}$, (2) or its square, with $\text{se}$ denoting standard error.
Suppose that rv’s X and Y are independent,
then so are
\[ g(X) \text{ and } h(Y) \]
(g, h measureable)

This is a further test that your residuals are white noise when acf “claims” they are.
Hence after fitting arma it is worth fitting garch.

**Acf’s**

![Exhibit 12.3 Sample ACF of Daily CREF Returns: 8/26/04 to 8/15/06](image)

```r
> acf(r.cref)
```
Exhibit 12.4  Sample PACF of Daily CREF Returns: 8/26/04 to 8/15/06

Exhibit 12.5  Sample ACF of the Absolute Daily CREF Returns

> pacf(r.cref)

> acf(abs(r.cref))
Exhibit 12.6  Sample PACF of the Absolute Daily CREF Returns

> pacf(abs(r.cref))

Exhibit 12.7  Sample ACF of the Squared Daily CREF Returns

> acf(r.cref^2)
McLeod-Li test:

Ljung-Box statistics with the autocorrelations of the squared data to detect for conditional heteroscedascity

(heteroscedasticity is the absence homoscedasticity)

Ljung-Box

\[ Q_\ast = n(n+2)\left( \frac{\hat{r}_1^2}{n-1} + \frac{\hat{r}_2^2}{n-2} + \cdots + \frac{\hat{r}_K^2}{n-K} \right) \]
skewed, thick tail
12.2 The ARCH(1) Model

the return series \{r_t\} is generated as follows:

\[ r_t = \sigma_{t|t-1}^{2} \epsilon_t \]
\[ \sigma_{t|t-1}^{2} = \omega + \alpha r_{t-1}^{2} \]
\[ E(r_t^2|r_{t-j}, j = 1, 2, \ldots) = E(\sigma_{t|t-1}^{2} \epsilon_t^2|r_r_{t-j}, j = 1, 2, \ldots) \]
\[ = \sigma_{t|t-1}^{2} E(\epsilon_t^2|r_{t-j}, j = 1, 2, \ldots) \]
\[ = \sigma_{t|t-1}^{2} E(\epsilon_t^2) \]
\[ = \sigma_{t|t-1}^{2} \]
The conditional variance is not directly observable.

Exhibit 12.11 Simulated ARCH(1) Model with $\omega = 0.01$ and $\alpha_1 = 0.9$

A main use of the ARCH model is to predict the future conditional variances. For example, one might be interested in forecasting the $h$-step-ahead conditional variance

$$\sigma^2_{t+h|t} = E(r^2_{t+h}|r_t, r_{t-1}, \ldots)$$  \hspace{1cm} (12.2.10)

It assumes a model

ARCH(q)

$$\sigma^2_{t|t-1} = \omega + \alpha_1 r^2_{t-1} + \alpha_2 r^2_{t-2} + \ldots + \alpha_q r^2_{t-q}$$

12.3 GARCH Models
GARCH(p,q)

\[ \sigma^2_{t|t-1} = \omega + \beta_1 \sigma^2_{t-1|t-2} + \cdots + \beta_p \sigma^2_{t-p|t-p-1} + \alpha_1 r^2_{t-1} + \alpha_2 r^2_{t-2} + \cdots + \alpha_q r^2_{t-q} \]

\[ (1 - \beta_1 B - \cdots - \beta_p B^p) \sigma^2_{t|t-1} = \omega + (\alpha_1 B + \cdots + \alpha_q B^q) r^2_t \]

In some of the literature, the notation GARCH(p,q) is written as GARCH(q,p); the orders are switched.

Two different sets of conventions are used in different software.
Exhibit 12.12 Simulated GARCH(1,1) Process

\begin{figure}
\centering
\includegraphics[width=\textwidth]{simulated_garch_plot}
\caption{Simulated GARCH(1,1) Process}
\end{figure}

\begin{verbatim}
> set.seed(1234567)
> garch11.sim=garch.sim(alpha=c(0.02, 0.05), beta=.9, n=500)
> plot(garch11.sim, type='l', ylab=expression(r[t]), xlab='t')
\end{verbatim}

Exhibit 12.13 Sample ACF of Simulated GARCH(1,1) Process

\begin{figure}
\centering
\includegraphics[width=\textwidth]{acf_plot}
\caption{Sample ACF of Simulated GARCH(1,1) Process}
\end{figure}

\begin{verbatim}
> acf(garch11.sim)
\end{verbatim}
Exhibit 12.14 Sample PACF of Simulated GARCH(1,1) Process

![Image of PACF](image1)

> pacf(garch11.sim)

Exhibit 12.16 Sample PACF of the Absolute Values of the Simulated GARCH(1,1) Process

![Image of PACF](image2)

> pacf(abs(garch11.sim))
For the CREF return data, we earlier identified either a GARCH(1,1) or GARCH(2,2) model. The AIC of the fitted GARCH(1,1) model is 969.6, whereas that of the GARCH(2,2) model is 970.3. Hence the GARCH(1,1) model provides a marginally better fit to the data. Maximum likelihood estimates of the fitted GARCH(1,1) model are reported in Exhibit 12.25.
12.5 Model Diagnostics

an adequate fit to the data. Recall that the standardized residuals are defined as

$$\hat{\epsilon}_t = \frac{r_t}{\hat{\sigma}_t}$$

which are approximately independently and identically distributed if the model is correctly specified. As in the case of model diagnostics for ARIMA models, the standardized residuals are very useful for checking the model specification. The

Exhibit 12.26 Standardized Residuals from the Fitted GARCH(1,1) Model of Daily CREF Returns

![Standardized Residuals Graph]
Exhibit 12.26 Standardized Residuals from the Fitted GARCH(1,1) Model of Daily CREF Returns

> plot(residuals(m1), type='h', ylab='Standardized Residuals')
Exhibit 12.27 QQ Normal Scores Plot of Standardized Residuals from the Fitted GARCH(1,1) Model of Daily CREF Returns

The portmanteau statistic equals

\[ n \sum_{k=1}^{m} \hat{\rho}_k^2 \]

statistic is approximately \( \chi^2 \) distributed with \( m \) hypothesis that the model is correctly specified.
Exhibit 12.28 Sample ACF of Squared Standardized Residuals from the GARCH(1,1) Model of the Daily CREF Returns

\begin{verbatim}
> acf(residuals(ml)^2, na.action=na.omit)
\end{verbatim}

Exhibit 12.29 Generalized Portmanteau Test p-Values for the Squared Standardized Residuals for the GARCH(1,1) Model of the Daily CREF Returns

\begin{verbatim}
> gBox(ml, method='squared')
\end{verbatim}
returns may be identified as a GARCH(1,2) process. However, the fitted GARCH(1,2) model to the CREF data did not improve the fit, as its AIC was 978.2—much higher than 969.6, that of the GARCH(1,1) model. Therefore, we conclude that the fitted GARCH(1,1) model provides a good fit to the CREF data.
For more detail see pages 457 – 460

12.6 Conditions for the Nonnegativity of the Conditional Variances

Because the conditional variance $\sigma^2_{t|t-1}$ must be nonnegative, the GARCH parameters are often constrained to be nonnegative. However, the nonnegativity parameter constraints need not be necessary for the nonnegativity of the conditional variances. This
12.7 Some Extensions of the GARCH Model

The GARCH model may be generalized in several directions. First, the GARCH model assumes that the conditional mean of the time series is zero. Even for financial time series, this strong assumption need not always hold. In the more general case, the conditional mean structure may be modeled by some ARMA(\(u, v\)) model, with the white noise term of the ARMA model modeled by some GARCH(\(p, q\)) model. Specifically, let \( \{ Y_t \} \) be a time series given by (now we switch to using the notation \( Y_t \) to denote a general time series)

\[
Y_t = \phi_1 Y_{t-1} + \cdots + \phi_u Y_{t-u} + \theta_0 + \theta_1 e_{t-1} + \cdots + \theta_v e_{t-v} \\
\epsilon_t = \sigma_{t-1} e_t \\
\sigma_{t-1}^2 = \sigma + \alpha_1 e_{t-1}^2 + \cdots + \alpha_q e_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2
\]  

(12.7.1)

and where we have used the plus convention in the MA parts of the model. The ARMA orders can be identified based on the time series \( \{ Y_t \} \), whereas the GARCH orders may be identified based on the squared residuals from the fitted ARMA model. Once the orders are identified, full maximum likelihood estimation for the ARMA + GARCH model can be carried out by maximizing the likelihood function as defined in Equation (12.4.4) on page 298 but with \( r_t \) there replaced by \( \epsilon_t \) that are recursively computed according to Equation (12.7.1). The maximum likelihood estimators of the ARMA parameters are approximately independent of their GARCH counterparts if the innovations \( \epsilon_t \) have a symmetric distribution (for example, a normal or \( t \)-distribution) and their
As an illustration for the ARIMA + GARCH model, we consider the daily USD/HKD (U.S. dollar to Hong Kong dollar) exchange rate from January 1, 2005 to March 7, 2006, altogether 431 days of data. The returns of the daily exchange rates are shown in Exhibit 12.33 and appear to be stationary, although volatility clustering is evident in the plot.

**Exhibit 12.33 Daily Returns of USD/HKD Exchange Rate: 1/1/05–3/7/06**

![Graph showing daily returns of USD/HKD exchange rate]

```r
> data(usd.hkd)
> plot(ts(usd.hkd$hkrate, freq=1), type='l', xlab='Day', ylab='Return')
```

It is interesting to note that the need for incorporating ARCH in the data is also supported by the McLeod-Li test applied to the residuals of the AR(1) + outlier model; see below for further discussion of the additive outlier. Exhibit 12.34 shows that the tests are all significant when the number of lags of the autocorrelations of the squared residuals ranges from 1 to 26, displaying strong evidence of conditional heteroscedasticity.
An AR(1) + GARCH(3,1) model was fitted to the (raw) return data with an additive outlier one day after July 22, 2005, the date when China revalued the yuan by 2.1% and adopted a floating-rate system for it. The outlier is shaded in gray in Exhibit 12.33. The

The AR + GARCH models partially reported in Exhibit 12.35 were fitted using the Proc Autoreg routine in the SAS software. \(^\dagger\) We used the default option of imposing that