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SOME RIVER WAVELETS

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SUMMARY

Wavelet analysis is described, and a Haar wavelet analysis is carried out, for time series data on the flow rate of the Nile River at Aswan and also on the stages of the Rio Negro at Manaus. A goal of the analysis is to present a wavelet analysis for some time series data particularly looking for jumps in mean level. The work begins with review of techniques for estimating mean levels in the presence of additive noise and then proceeds to the particular case of wavelets and the construction of so-called 'improved estimates' by shrinkage. The results of the analyses are consistent with earlier ones.

KEY WORDS Change Mean level Nile River flow Rio Negro stages Shrinkage Smoothing Wavelet

1. INTRODUCTION

Time series data are fundamental to discussions concerning the environment in fields like climatology, ecology, hydrology and oceanography. Questions arise such as: Is there a smooth trend? Has there been a change in mean level? Have there been several changes? The top panels of Figures 1 and 2 give graphs, respectively, of the seasonally adjusted monthly stages of the Rio Negro at Manaus, Brazil, from 1903 to 1992 and of the annual Nile River flow at Aswan from 1871 to 1970. The work will be done with these examples. One can wonder whether there have been noteworthy trends or changes in the mean levels of these series. The data are discussed in MacNeill *et al.*[1] in the case of the Nile and Sternberg[2] in the case of the Rio Negro. Because the terrain is flat, the Rio Negro values are a proxy for the Amazon River.

Sometimes, scientific queries may be related to analytic questions concerning the mean function of a time series model. One goal of the paper is to bring out the simplicity of Haar wavelet analysis in producing an estimate of a mean function and to show that the methodology is in parallel with the common techniques of running means and kernel smoothers. Contributions of the paper include the suggestion of a wavelet estimate for the case of additive stationary errors, the construction of uncertainty limits for the estimate and results from employing a particular shrinker.

The paper begins with review of methods for estimating mean functions, including wavelet analysis, then that technique is employed in an examination of the two river data sets. Consistent with early studies, there is evidence for a change in the mean level of the Nile discharge after 1900, but little evidence of a change in the Rio Negro. In conclusion, it is found that the wavelet technique has favourable prospects in the field of environmental time series analysis and that it can take simple plausible form.

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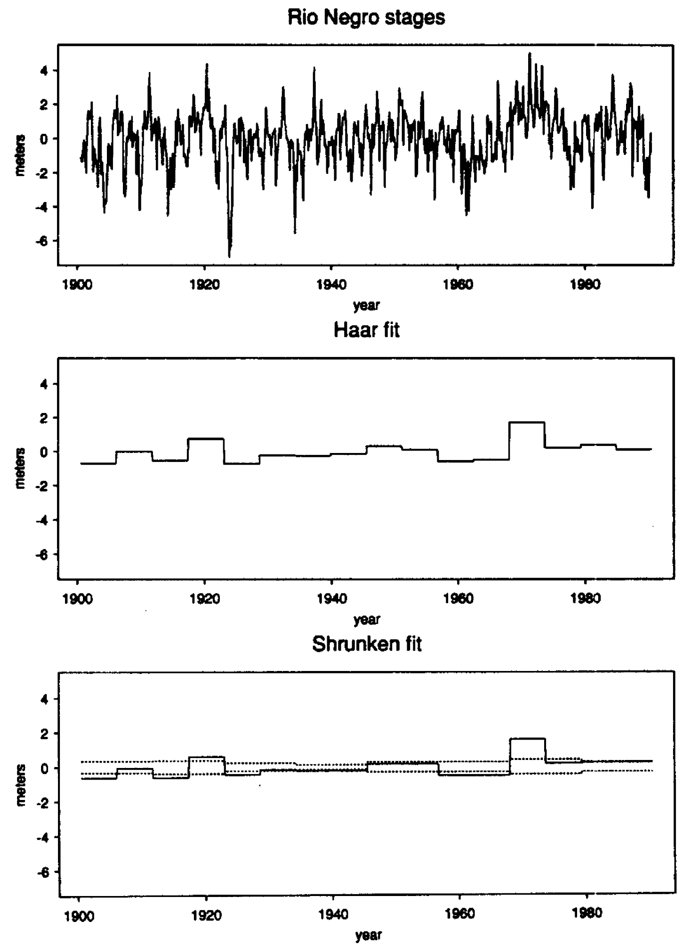

Figure 1. The top panel is obtained from daily Rio Negro stages by computing the monthly averages, then removing the overall monthly means to get a seasonally adjusted monthly series. The middle panel is the naive Haar estimate (15) with $J = 3$. The bottom panel provides the wavelet estimate (19) employing the multiplier function (17). The dashed lines give approximate ± 2 standard error limits about the overall mean level

2. MODELLING MEAN FUNCTIONS

Consider the model

$$Y(t) = S(t) + E(t) \tag{1}$$

$t = 0, \pm 1, \pm 2, \ldots$ with $S(\cdot)$ a deterministic signal and $E(\cdot)$ a stationary noise, that is $E\{Y(t)\} = S(t)$ is the mean level of the series $Y(\cdot)$ at time t. Quite a variety of different procedures have been proposed for estimating $S(t)$ given data $Y(t)$, $t = 0, \ldots, T-1$. These methods can be linear or non-linear and parametric or non-parametric. Some of the procedures are described next.

2.1. Parametric procedures

To begin, consider the case of a finite parameter linear model, such as

$$E\{Y(t)\} = S(t|\alpha) = \alpha_1 g_1(t) + \cdots + \alpha_J g_J(t) \tag{2}$$

with J known and the $g_1(\cdot) \ldots, g_J(\cdot)$ given functions (they could be polynomial or trigonometric). Questions of trend and change might be formulated as a hypothesis that some of the α_j are 0. The parameter α may be estimated, for example, by ordinary least squares.

Supposing that the data are available for $t = 0, \ldots, T-1$ and that the $g_k(t)$ satisfy

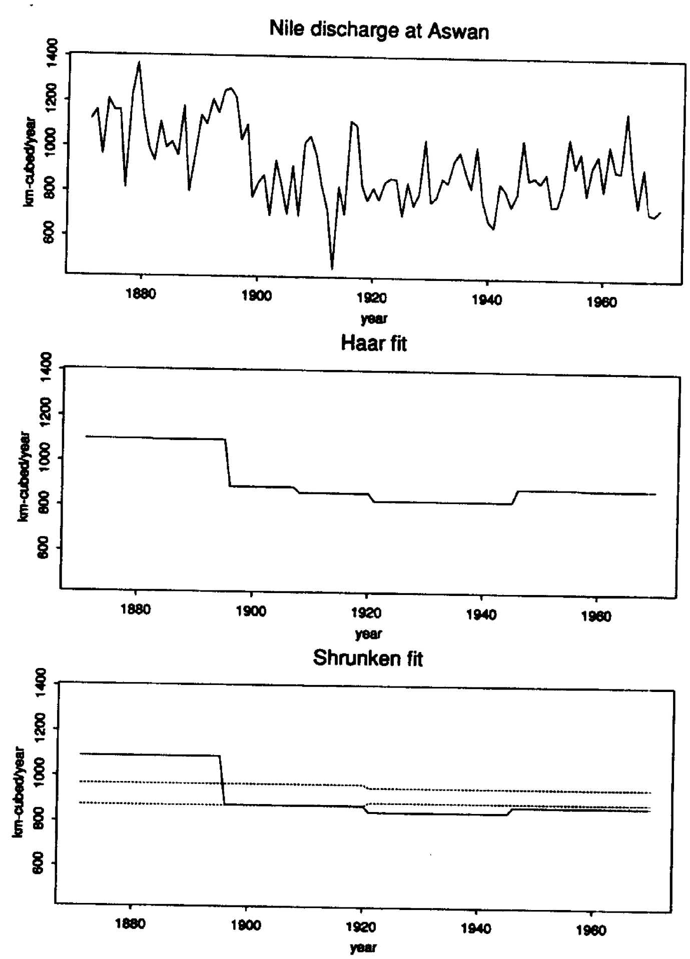

Figure 2. The top panel graphs the annual discharge of the Nile at Aswan for the period 1871 to 1970. The middle panel is the naive Haar estimate (15) with $J = 3$. The bottom panel provides the wavelet estimate (19) employing the multiplier function (17). The dashed lines give approximate ± 2 standard error limits

Grenander's conditions such as, for some N_T

$$\lim_{T\to\infty} \frac{1}{N_T} \sum_{t=0}^{T-1} g_j(t+u) g_k(t) = m_{jk}(u)$$

the large sample distribution of the ordinary and of best linear unbiased least squares estimates may be determined (see Grenander and Rosenblatt,[3] Hannan,[4] Anderson,[5] Brillinger.[6] The ordinary least squares estimate is asymptotically efficient in important cases. Confidence limits about a fitted level

$$\hat{S}(t) = S(t|\hat{\alpha}) = \hat{\alpha}_1 g_1(t) + \cdots + \hat{\alpha}_J g_J(t)$$

may be set down making use of the asymptotic distribution.

Hannan[7,8] further considers the case of $g_j^T(t)$, where the regressors depend on T, with analogues of Grenander's conditions holding. Results are also available for the case of non-linear regression, the function $S(t|\theta)$ being known up to a parameter θ. Exponential trends and hidden periodicities are included within these latter models. Limits of Grenander type,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} S(t+u|\theta^1) S(t|\theta^2)$$

are taken to exist. Asymptotic distributions may be derived, questions of efficiency addressed and hypotheses that specified components of θ are 0 may be examined, see Hannan,[9] Robinson,[10] Gallant and Goebel.[11]

2.2. Non-parametric procedures

In the case that the mean function $S(t)$ is smooth, one can consider its estimation by a running mean (or kernel smoother), i.e. an expression of the form

$$\hat{S}(t) = \sum_s w_b(t-s) Y(s) / \sum_s w_b(t-s) \tag{3}$$

where the kernel, $w_b(\cdot)$, has binwidth b. (In the case of a running mean, $w(\cdot)$ would correspond to a uniform density.) This type of estimate has been computed for decades, see for example Macaulay.[12] In the case of known b, the estimate (3) is linear so various approximate distribution results may be developed directly. See Brillinger,[13] Bloomfield and Nychka[14] for applications. Härdle and Tuan[15] present results including robust procedures. The problem of estimating b is considered in Chiu,[16] Hart,[17] Altman.[18] The optimal b is determined in Truong,[19] Truong and Stone.[20]

The just described procedures employ uniform binwidth for all t. Variable binwidth smoothers have been proposed on occasion, see Hastie and Tibshirani.[21] The wavelet estimates have this variable character.

Further approaches to the estimation of the function $S(\cdot)$ include: orthogonal series expansions;[22,23] smoothness priors;[24–26] regression;[27] splines.[28,29] There is general discussion of smoothers in Hastie and Tibshirani.[21]

3. WAVELETS

Wavelets are a contemporary tool for function approximation. They are competitors/collaborators with traditional Fourier analysis and other orthogonal function expansions as above. In particular they are useful for handling localized behaviour, discontinuities, and scale and shift transformations.

3.1 Introduction

Wavelet analyses correspond to particular orthonormal series expansions. One starts with a function $\phi(\cdot)$ satisfying a so-called scaling identity

$$\phi(x) = \sum_k c_k \phi(2x - k)$$

such that the $\phi_k(x) = \phi(x - k)$, $k = -\infty, \ldots, \infty$ are orthogonal. Then one obtains

$$\psi(x) = \sum_k (-1)^k c_{-k+1} \phi(2x - k)$$

and an associated orthonormal family

$$\psi_{jk}(x) = 2^{j/2} \psi(2^j x - k). \tag{4}$$

For a square-integrable function $h(x)$, one has the orthogonal series expansion

$$h(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \beta_{jk} \psi_{jk}(x) \tag{5}$$

with

$$\beta_{jk} = \int \psi_{jk}(x) h(x) \mathrm{d}x. \tag{6}$$

General references include Daubechies,[30] Walter,[31,32] Strichartz,[33] Benedetto and Frazier.[34]

Expression (5) may be usefully written

$$h(x) = h_0(x) + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_{jk} \psi_{jk}(x) \tag{7}$$

where

$$h_0(x) = \sum_{k=-\infty}^{\infty} \alpha_k \phi_k(x) \tag{8}$$

with

$$\alpha_k = \int \phi_k(x) h(x) \mathrm{d}x. \tag{9}$$

In the case of a function with discontinuities, a naive wavelet analysis may be suitable, namely Haar analysis. This is based on the particular functions

$$\phi(x) = \begin{matrix} 1 & 0 \leqslant x < 1 \\ 0 & \text{otherwise} \end{matrix}$$

satisfying $\phi(x) = \phi(2x) + \phi(2x - 1)$ and

$$\psi(x) = \begin{matrix} 1 & 0 \leqslant x < 1/2 \\ -1 & 1/2 \leqslant x < 1 \\ 0 & \text{otherwise} \end{matrix} \tag{10}$$

satisfying $\psi(x) = \phi(2x) - \phi(2x-1)$. Expressions (6) and (9) take particularly simple form in this case.

3.2. Some specifics

General motivation and discussion may be found in Bock.[35] Suppose that data, $Y(t)$, $t = 0, \ldots, T-1$ are available. In developing estimates and approximations it is convenient to write $S(t) = h(t/T)$ with $h(x) = 0$ outside (0,1). An elementary estimate of $S(t)$ suggested by (8) and (9) is

$$\hat{S}(t) = \sum_{k=-\infty}^{\infty} \hat{\alpha}_k \phi_k(t/T) + \sum_{j=0}^{J} \sum_{k=-\infty}^{\infty} \hat{\beta}_{jk} \psi_{jk}(t/T) \tag{11}$$

with J representing the finest scale of interest and with

$$\hat{\alpha}_k = \frac{1}{T} \sum_t \phi_k(t/T) Y(t) \tag{12}$$

$$\hat{\beta}_{jk} = \frac{1}{T} \sum_t \psi_{jk}(t/T) Y(t). \tag{13}$$

(Note that, despite its appearance, (11) involves but a finite number of terms.)

In the case of the Haar wavelet, things simplify. The only $\hat{\alpha}$ is $\hat{\alpha}_0$ and it is the mean of the Y's. The form of $\hat{\beta}_{jk}$ is

$$\hat{\beta}_{jk} = \frac{2^{j/2}}{T} \left[\sum_t{}' Y(t) - \sum_t{}'' Y(t) \right] \tag{14}$$

where $\sum'$ is over $0 \leqslant 2^j t/T - k < \frac{1}{2}$ and $\sum''$ is over $\frac{1}{2} \leqslant 2^j t/T - k < 1$. Computing such local means, in either a smoothing or a search for change-points, seems intuitively reasonable. The estimate (11) is simply

$$\hat{S}_0(t) = \hat{\alpha}_0 + \sum_{j=0}^{J} \sum_{k=0}^{2^j - 1} \hat{\beta}_{jk} 2^{j/2} \psi\left(2^j \frac{t}{T} - k\right) \tag{15}$$

i.e. a linear combination of step functions.

The statistics (11), (12) and (13) are linear in the Y's, hence sampling properties are directly available, e.g. large sample variances and distributions. In the case of (13) for example

$$\operatorname{var} \hat{\beta} = \int_{-\pi}^{\pi} |\Psi^{\mathrm{T}}(\lambda)|^2 f_{\mathrm{EE}}(\lambda) \mathrm{d}\lambda \approx \frac{1}{T} \int_0^1 \psi(x)^2 \mathrm{d}x 2\pi f_{\mathrm{EE}}(0) \tag{16}$$

with $f_{\mathrm{EE}}(\lambda)$ the power spectrum of the noise $E(\cdot)$ at frequency λ and

$$\Psi^{\mathrm{T}}(\lambda) = \frac{1}{T} \sum_{t=0}^{T-1} e^{-i\lambda t} \psi(t/T).$$

An estimate of (16) will be needed to form the estimates of the next section. It and an estimate of the variance of (15) will be developed in the Appendix.

3.3 Shrinkage estimates

Shrinkage is basic to the computation of wavelet estimates, see Donoho and Johnstone,[36] Donoho[37] and Hall and Patil.[38] The suggestion to use shrinkers or multipliers to 'improve' estimates has been around in statistics for many years, see Lemmer.[39] There is an early harmonic synthesis application in Blow and Crick,[40] concerned with crystal imaging.

A shrinkage regression estimate involves regression coefficients $\hat{\beta}$ being multiplied by factors between 0 and 1 depending on their individual uncertainty. For example, $\hat{\beta}$ may be shrunk to

$$w(\hat{\beta}/s)\hat{\beta}$$

where s is an estimate of its standard error and $w(\cdot)$ is a function such that $w(u) \approx 1$ for large $|u|$ and ≈ 0 for small $|u|$. Tukey,[41] for example, proposes

$$w(u) = (1 - 1/u^2)_+. \tag{17}$$

It may be noted that this multiplier weights to 0 all terms where $|\hat{\beta}|$ is less than its standard error. Saleh and Han[42] take $w(\cdot)$ to be a function of a test statistic for the hypothesis that β is 0.

In the wavelet case, one can consider the shrinkage estimator

$$\hat{S}(t) = \sum_{k=-\infty}^{\infty} \hat{\alpha}_k \phi_k(t/T) + \sum_{j=0}^{J} \sum_{k=-\infty}^{\infty} w(\hat{\beta}_{jk}/s_{jk})\hat{\beta}_{jk}\psi_{jk}(t/T) \tag{18}$$

where s_{jk}^2 is an estimate of the variance of $\hat{\beta}_{jk}$. Donoho and Johnstone[36] and Hall and Patil[38] suggest some multipliers, of the form $w(\hat{\beta}/sc_T)$ with, e.g. $c_T = \sqrt{(2\log T)}$.

In the Haar case, (18) becomes

$$\hat{S}(t) = \hat{\alpha}_0 + \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} \hat{\beta}_{jk} w_{jk} 2^{j/2} \psi\left(2^j \frac{t}{T} - 1\right) \tag{19}$$

writing w_{jk} for the multiplier. The construction of an estimate of the standard error of (19) is indicated in the Appendix.

4. EXAMPLES

The first example is based on the stages of the Amazon River at Manaus, 1903–1992. Monthly values are employed, having reduced the seasonal effect by removing overall monthly means. (In the case that a trend is present, the seasonal will not be removed completely.) The length of the series is $T = 1080$. The first panel of Figure 1 plots the seasonally adjusted values. The second gives the Haar fit (15), taking the finest level of detail to be $J = 3$. This graph shows how the estimate corresponds to the series being divided into 16 contiguous segments, then the mean level being estimated throughout a segment by the average value of the data in the segment, see Brillinger.[43] The final panel provides the shrunken estimate (19), with the multiplier $w(u) = (1 - 1/u^2)_+$. One sees that a number of the steps of the middle panel are gone and that the values have been shrunk towards the middle. The dashed lines of the figure, whose computation is described in the Appendix, give approximate marginal ± 2 standard error limits about the overall mean level, computed using (20). An increase in average level in the years around 1970 stands out.

The second example concerns Nile River discharge at Aswan, Egypt. This series has been a testbed for change-point techniques in the past. The data themselves are listed in Cobb.[44] They are annual July–June flows from 1871 to 1970. The $T = 100$ values are graphed in Figure 2. The simple Haar estimate (15), is graphed in the middle panel taking $J = 2$. The estimate (19) is graphed in the bottom panel, again using the multiplier (17). The dashed curve gives approximate marginal ± 2 standard error limits about the overall mean level. One sees that the first quarter of the fit is well outside the limits. This result is consistent with previous studies.[1] These authors note that a dam was built at Aswan in the period 1899 to 1902.

6. DISCUSSION

Wavelets estimates have been found to take on a simple and natural form in the Haar case. The detail of the estimate is seen to vary with t in contrast with the usual running mean or kernel estimates. The profile of the estimate in this Haar case is such as to highlight possible jumps in the mean level. As set out in the Appendix, the analyses have involved both wavelet and Fourier analyses, the former to obtain an estimate of the mean level, the latter to estimate its uncertainty.

The confidence bounds in the figures are marginal. Approximate simultaneous bounds could be constructed in a manner extending Bjerve *et al.*[45] and Eubank and Speckman.[46]

Other approaches remain, Step functions of unknown step points might be fit. Donoho[47] presents a procedure that picks up general change-points, not those simply of the form $k/2^j$ as is the case here. Finally, experience with different multipliers and the accuracies of the approximations made needs to be gained.

APPENDIX

To form the shrunken estimate (18) one needs estimates of the var $\hat{\beta}$ given by (16). The spectrum value $f_{EE}(0)$ needs to be estimated. The method employed in the data analyses is to compute the simple fit, $\hat{S}_0(t)$ of (15), then the residuals, $\hat{E}(t) = Y(t) - \hat{S}_0(t)$ and then to average the periodograms near 0 (see Brillinger[6]).

To determine a variance estimate for $\hat{S}(t) - \hat{\alpha}_0$ of (15) one needs an expression for

$$\operatorname{cov}\{\hat{\beta}_{jk}, \hat{\beta}_{j'k'}\}.$$

One such is

$$\int_{-\pi}^{\pi} \Psi_{jk}^T(\lambda)\Psi_{j'k'}^T(\lambda) f_{EE}(\lambda)\mathrm{d}\lambda \approx \frac{1}{T}\int_0^1 \psi_{jk}(x)\psi_{j'k'}(x)\mathrm{d}x 2\pi f_{EE}(0)$$

giving (16) when $(j,k) = (j',k')$ and 0 otherwise, following the orthogonality of the ψ's.

Inserting multipliers, as in (16), complicates the computation of standard errors, but it is hoped that this effect is secondary, as is the case in some similar situations. In the computations the multipliers are treated as constants, and the variance estimate taken to be

$$\sum_{j=0}^{J}\sum_{k=0}^{2^j-1} \psi_{jk}\left(\frac{t}{T}\right)^2 w_{jk}^2 \frac{1}{T}\int_0^1 \psi_{jk}(x)^2 \mathrm{d}x 2\pi \hat{f}_{EE}(0) \tag{20}$$

with $w_{jk} = w(\hat{\beta}_{jk}/s_{jk})$.

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