Dr D. R. Brillinger: Among the problems raised in the last section of Mr Sprent's paper and noted by the previous speakers is that of setting confidence limits for his proposed estimates. Given the nature of these estimates, that is, that they are evidently best calculated by means of a computer and in addition no simple formula for their variance has been presented, this would appear to be an ideal place to use Professor J. W. Tukey's jack-knife procedure.

The jack-knife proceeds as follows: given $n = rs$ observations, divide them into r groups of s. Let q denote an estimate of a parameter of interest based on all n observations and let q_i denote an estimate based on the $s(r-1)$ observations obtained by omitting the ith group. Define the pseudo-values $q_{(i)} = rq-(r-1)\,q_i$. The estimate $\bar{q} = \Sigma q_{(i)}/r$ eliminates bias of order $1/n$ in the manner of Quenouille (1956), while, more importantly,

$$S^2 = \frac{1}{r(r-1)}\Sigma(q_{(i)}-\bar{q})^2 = \frac{r-1}{r}(\Sigma q_i^2-(\Sigma q_i)^2/r)$$

provides an estimate of the variances of q and $\bar{q}$. Confidence limits may now be set using the t distribution on $r-1$ degrees of freedom.

Professor Tukey has justified the use of the estimate S^2 for the class of sequilinear estimates. Justifications of the use of S^2 and indicating an alternate estimator run as follows: let θ denote the parameter of interest. Suppose

$$\text{(i)}\quad \operatorname{ave} q = \theta+o(1/s),$$

$$\text{(ii)}\quad \operatorname{ave} q_i = \theta+o(1/s),$$

$$\text{(iii)}\quad \operatorname{var} q = \sigma^2/n+o(1/s),$$

$$\text{(iv)}\quad \operatorname{var} q_i = \sigma^2/s(r-1)+o(1/s).$$

We see that to $o(1/s)$, $\operatorname{ave} q_i^2 = \theta^2+\sigma^2/s(r-1)$, $\operatorname{ave} q^2 = \theta^2+\sigma^2/n$ and therefore

$$\operatorname{ave}\{(r-1)/r\}(\Sigma q_i^2-rq^2) = \operatorname{var} q+o(1/s).$$

Alternatively, if we have (iv) and

$$\text{(v)}\quad \operatorname{corr}\{q_i, q_j\} = (r-2)/(r-1)+o(1/s) \quad (i \neq j),$$

then

$$\operatorname{ave} S^2 = \sigma^2/n+o(1/s) = \operatorname{var} q.$$

(The motivation for (v) is the fact that q_i and q_j have $r-2$ groups in common out of the $r-1$ available.)

Note that we have made no assumption concerning the independence of the observations. Also note that the assumptions (i) to (iv) indicate properties that we seek to achieve by the division into groups and the form of q_i. In connection with this last, (iv) may be achieved on occasion by applying a missing-values technique. That is, we have taken n observations, but s are to be looked upon as missing, leading to the estimate q_i. We have taken q_i based on $s(r-1)$ rather than s observations in the hope that the remainder terms in (ii) and (iv) are smaller.

Returning to the specific estimates of Mr Sprent's paper, by the imposition of conditions on Σ_0 we can hope to satisfy (i) to (iv). The sort of conditions I have in mind are that the $(\epsilon_{xi}, \epsilon_{yi})$ come from a stationary process or that they have independence properties in the manner of the example in Section 2 of the paper.

In conclusion, let me note that the jack-knife procedure extends in an immediate manner to the multiple parameter situation. Two recent references devoted to the jack-knife are Miller (1964) and Brillinger (1964).

The following written contribution was received after the meeting:

Mr P. R. Fisk: Mr Sprent appears to say, in Section 1, that a constraint on β to determine a constant of proportionality is sufficient to obtain a unique solution. This is