STAT 150 HOMEWORK #1

SPRING 2024

Due Friday, Jan 26, at 11:59 PM on Gradescope.

Note that there are Exercises and Problems in Pinsky and Karlin. Make sure you read the homework carefully to find the assigned question.

1. Pinsky and Karlin, Problem 1.3.11
2. Pinsky and Karlin, Problem 1.4.3
3. Pinsky and Karlin, Problem 1.5.7
4. Pinsky and Karlin, Problem 1.5.8
5. Pinsky and Karlin, Problem 2.1.2
6. Pinsky and Karlin, Problem 2.1.9
7. Let X be random variable taking values in $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Prove that $X \sim \text{Poisson}(\lambda)$ if and only if for every function $f : \mathbb{N}_0 \to \mathbb{R}_+$,
$$\mathbb{E}[Xf(X)] = \lambda \mathbb{E}[f(X+1)].$$
8. Let X be a random variable. Recall that the moment generating function (or MGF for short) $M_X(t)$ of X is the function $M_X : \mathbb{R} \to [0, \infty]$ defined by $t \mapsto \mathbb{E}[e^{tX}]$. Now suppose that $X \sim \text{Gamma}(\alpha, \lambda)$, where $\alpha, \lambda > 0$. Prove that
$$M_X(t) = \begin{cases} \left(\dfrac{\lambda}{\lambda - t}\right)^{\alpha} & \text{if } t < \lambda; \\ \infty & \text{if } t \geq \lambda. \end{cases}$$
9. Let X be a random variable with finite variance. Prove that the mean $\mu = \mathbb{E}[X]$ is the unique minimizer of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(r) = \mathbb{E}\left[|X - r|^2\right]$. In other words, prove that $f(r) \geq f(\mu)$ for any $r \in \mathbb{R}$ with equality if and only if $r = \mu$.
10. Let X be a random variable. We say that $m \in \mathbb{R}$ is a *median* of X if
$$\min\{\mathbb{P}(X \leq m), \mathbb{P}(X \geq m)\} \geq \frac{1}{2}.$$
You may assume that a median always exists (for fun, you can try to prove this).
(a) Is a median necessarily unique? Prove or provide a counterexample.
(b) Suppose that m is a median of X. Prove that
$$\mathbb{P}(X \geq m + \varepsilon) \leq \mathbb{P}(X \leq m + \varepsilon)$$

for any $\varepsilon > 0$. Think about why this should be true intuitively.

(c) Assume that $\mathbb{E}[|X|] < \infty$. Prove that a median minimizes the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(r) = \mathbb{E}\left[|X - r|\right]$ (note the contrast to problem 8). Hint: you may use the fact that

$$\mathbb{E}[Y] = \int_0^\infty \mathbb{P}(Y \geq t)\, dt$$

for any non-negative random variable Y. Apply this to compare $\mathbb{E}\left[|X - r|\right]$ and $\mathbb{E}\left[|X - m|\right]$ with the help of part (b). You may also assume that $r > m$ (ask yourself why you can make this assumption though).