Spectral asymptotics for contracted tensor ensembles

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Joint work with Jorge Garza-Vargas

- Setting: d-th order N-dimensional real symmetric tensors $T_{d,N} = (T_{d,N}(k_1, \dots, k_d))_{(k_1,\dots,k_d) \in [N]^d} \in \mathcal{S}_{d,N} \subseteq \mathbb{R}^{N^d}$,

$$T_{d,N}(k_1, \dots, k_d) = T_{d,N}(k_{\sigma(1)}, \dots, k_{\sigma(d)})$$

- Example: $d = 3$

- Basic question: how does the randomness of $T_{d,N}$ behave under repeated contractions?

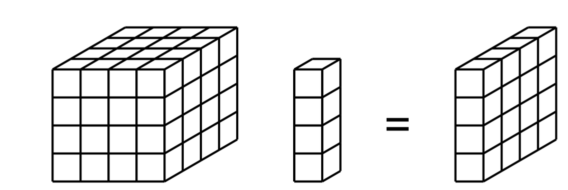
- For $p \leq d$ and vectors $v_1, \dots, v_p \in \mathbb{R}^N$, we define the contracted tensor

$$T_{d,N}[v_1 \otimes \cdots \otimes v_p] \in \mathcal{S}_{d-p,N}$$

by

$$T_{d,N}[v_1 \otimes \cdots \otimes v_p](k_1, \dots, k_{d-p}) = \sum_{\ell_1,\dots,\ell_p} T_{d,N}(k_1, \dots, k_{d-p}, \ell_1, \dots, \ell_p)\, v_1(\ell_1) \cdots v_p(\ell_p)$$

- Example:



- Example: $T_{2,N}$ Wigner, $v_N^{(1)}, v_N^{(2)}, w_N^{(1)}, w_N^{(2)} \in S^{N-1}$

$\sigma^2 = 1$ on diagonal $\quad \|\cdot\|_\infty = o(1)$

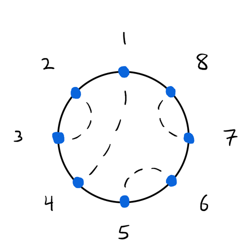
$$\begin{pmatrix} T_{2,N}[v_N^{(1)} \otimes v_N^{(2)}] \\ T_{2,N}[w_N^{(1)} \otimes w_N^{(2)}] \end{pmatrix} \xrightarrow{d} \mathcal{N}(\hat{0}, \mathcal{K}),$$

$$\mathcal{K} = \lim_{N \to \infty} \begin{pmatrix} \langle v_N^{(1)} \odot v_N^{(2)}, v_N^{(1)} \odot v_N^{(2)} \rangle & \langle v_N^{(1)} \odot v_N^{(2)}, w_N^{(1)} \odot w_N^{(2)} \rangle \\ \langle w_N^{(1)} \odot w_N^{(2)}, v_N^{(1)} \odot v_N^{(2)} \rangle & \langle w_N^{(1)} \odot w_N^{(2)}, w_N^{(1)} \odot w_N^{(2)} \rangle \end{pmatrix}$$

$$u_1 \odot \cdots \odot u_d = \frac{1}{d!} \sum_{\sigma \in S_d} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \in \mathcal{S}_{d,N}$$

- (Voiculescu) Independent Wigner matrices $(W_N^{(i)})_{i \in I}$ are asymptotically free, converge in distribution to a multivariate semicircle $SC(\hat{0}, \underbrace{\tfrac{1}{2} \mathrm{Id}_{\#(I)}}_{\mathcal{K}})$:

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \mathrm{Tr}[W_N^{(i_1)} \cdots W_N^{(i_m)}]\right] = \sum_{\underbrace{\pi \in NC_2(m)}_{\text{non-crossing}}} \prod_{\{j,k\} \in \pi} \mathcal{K}(i_j, i_k)$$



- Back to tensors: if $T_{d,N}$ is a random symmetric tensor, the contracted tensor ensemble (GCC21) is the family of random matrices $\{T_{d,N}[u^{\otimes d-2}]\}_{u \in S^{N-1}}$
- What kind of randomness?

A canonical distribution:

$$\text{(GOTE)} \qquad \frac{1}{Z_{d,N}} e^{-\|H\|_F^2/2}\, dH$$

- (GCC21) For any sequence of unit vectors $u_N \in S^{N-1}$, the empirical spectral distribution of $W_N = \frac{1}{\sqrt{N}} T_{3,N}[u_N]$ (GOTE) converges weakly a.s. to the semicircle distribution with $\sigma^2 = \frac{1}{6}$
- In general, W_N is not a Wigner matrix:

$$T_{3,N}[u_N](j,k) = \sum_{\ell} T_{3,N}(j,k,\ell)\, u_N(\ell)$$

- Proof relies on Stein's method and $d = 3$

- Q1: what about higher order $d \geq 4$?

$$T_{3,N}[u_N](j,k) = \sum_{\ell} T_{3,N}(j,k,\ell)\, u_N(\ell)$$

$$T_{4,N}[u_N^{\otimes 2}](j,k) = \sum_{\ell_1,\ell_2} T_{4,N}(j,k,\ell_1,\ell_2)\, u_N(\ell_1) u_N(\ell_2)$$

- Q2: universality for general distributions?
- Q3: general contractions $u_N^{(1)} \otimes \cdots \otimes u_N^{(d-2)} \neq u_N^{\otimes d-2}$?
- Q4: joint behavior of $\{T_{d,N}[u_N^{(1)} \otimes \cdots \otimes u_N^{(d-2)}]\}_{u_N^{(j)} \in S^{N-1}}$?

- For our purposes, a Wigner tensor is a random real symmetric tensor $T_{d,N}$ such that:

(1) upper triangular entries are independent;

(2) off-diagonal entries are centered;

(2′) entries with $\#(k_1, \dots, k_d) \geq 3$ have variance $\boxed{\frac{1}{Z_{d,N}} e^{-\|H\|_F^2/2}\, dH} \rightarrow \binom{d}{b_1, \dots, b_N}^{-1}$;

(3) for any m,

$$\sup_{N \in \mathbb{N}} \sup_{\vec{k}} \mathbb{E}[|T_{d,N}(k_1, \dots, k_d)|^m] = C_m < \infty$$

- For vectors $u_1, \dots, u_d \in \mathbb{R}^N$, define the symmetrization

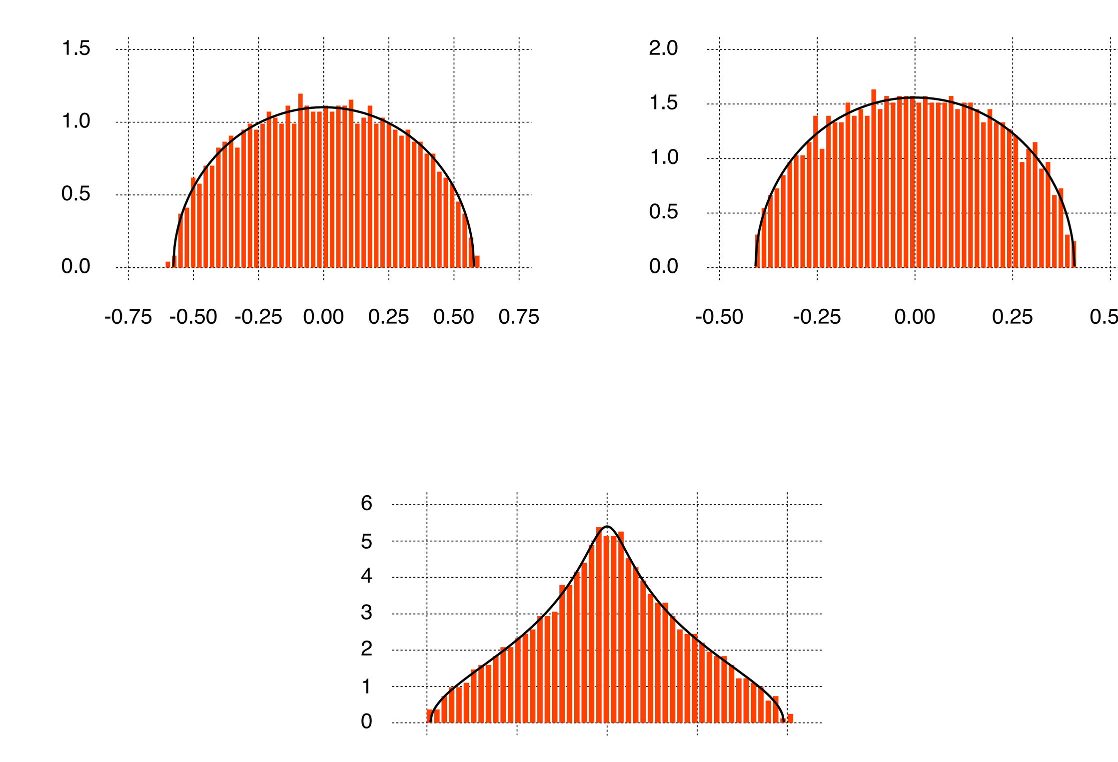
$$u_1 \odot \cdots \odot u_d = \frac{1}{d!} \sum_{\sigma \in S_d} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \in \mathcal{S}_{d-p,N}.$$

- For any sequence of families of unit vectors $\{u_N^{(i,j)}\}_{i \in I,\, j \in [d-2]}$, let $\mathcal{K}^{(N)} = (\mathcal{K}^{(N)}(i,i'))_{i,i' \in I}$ be the rescaled Gram matrix of the symmetrizations:

$$\mathcal{K}^{(N)}(i,i') = \frac{1}{d(d-1)} \left\langle u_N^{(i,1)} \odot \cdots \odot u_N^{(i,d)},\ u_N^{(i',1)} \odot \cdots \odot u_N^{(i',d)} \right\rangle.$$

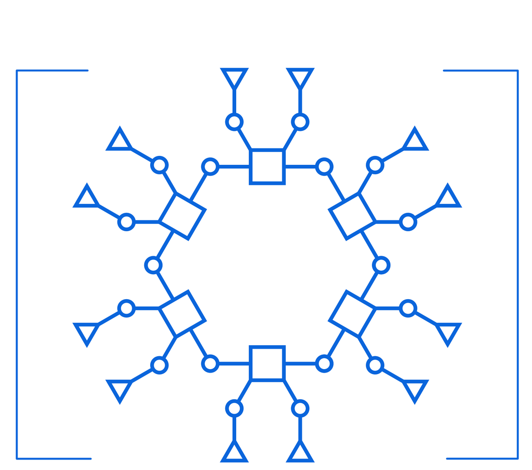
- (AGV21) $(W_N^{(i)})_{i \in I} = \left(\frac{1}{\sqrt{N}} T_{d,N}[u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)}]\right)_{i \in I} \approx SC(\hat{0}, \mathcal{K}_N)$: for any finite subset $I_0 \subseteq I$, exponent rate M, moment threshold m_0, and error $\varepsilon > 0$, there is a constant $C = C(d, \#(I_0), M, m_0, \varepsilon)$ s.t.

$$\mathbb{P}\left[\max_{\substack{m \leq m_0 \\ i_1,\dots,i_m \in I_0}} \left| \frac{1}{N} \mathrm{Tr}[W_N^{(i_1)} \cdots W_N^{(i_m)}] - \sum_{\pi \in NC_2(m)} \prod_{\{j,k\} \in \pi} \mathcal{K}_N(i_j, i_k) \right| > \varepsilon \right] < \frac{C}{N^M}$$



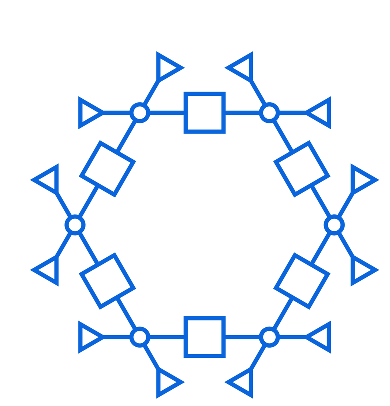
- What is the complication? Too many vertices!

$$\sum_{\pi \in P(V)} \sum_{\substack{\phi: V^\pi \to [N] \\ \text{injective}}}$$



$$\mathrm{Tr}[W_N^{(i_1)} \cdots W_N^{(i_6)}]$$

- What is the solution? Unit vectors!



- Naive bound: $O(N^2)$ / True bound: $O(N^{-4})$

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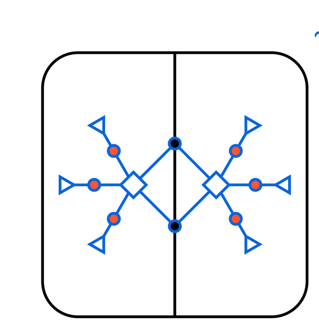
inner/outer

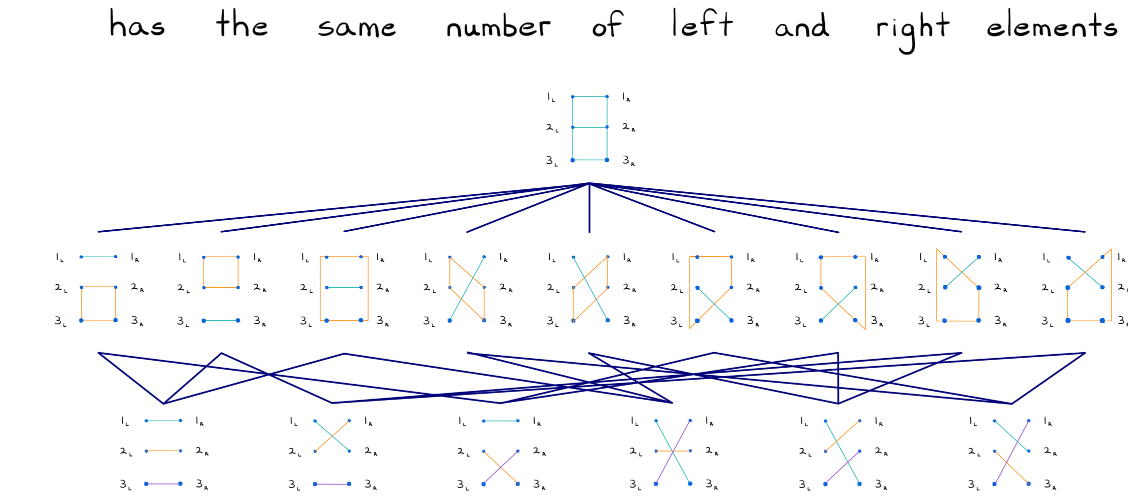
$$\sum_{\substack{\pi \in P(V) \\ \text{s.t.} \\ \pi = \pi|_{V_{\text{inner}}} \cup \pi|_{V_{\text{outer}}}}} \sum_{\substack{\phi: V^\pi \to [N] \\ \text{injective}}}$$



$$= \frac{1}{d!} \sum_{\sigma \in S_{d-2}} \prod_{j=1}^{d-2} \langle u_N^{(i,j)}, u_N^{(i',\sigma(j))} \rangle$$

(2′) entries with $\#(k_1, \dots, k_d) \geq 3$ have variance $\binom{d}{b_1, \dots, b_N}^{-1}$;

- A uniform block permutation of $[n]$ is a partition $\pi \in P([n]_L \sqcup [n]_R)$ such that each block $B \in \pi$ has the same number of left and right elements:



$(UBP(3), \leq)$