Spectral asymptotics for contracted tensor ensembles

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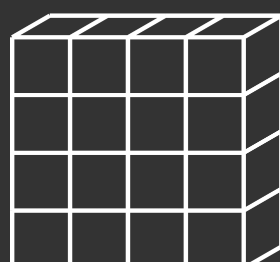
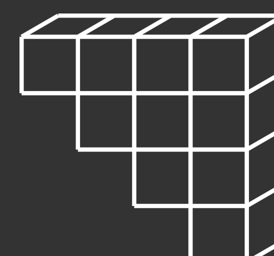
Joint work with Jorge Garza-Vargas

- Setting: d-th order N-dimensional real square symmetric tensors $T_{d,N} = \left(T_{d,N}(k_1, \dots, k_d)\right)_{(k_1,\dots,k_d) \in [N]^d} \in \mathcal{S}_{d,N} \subseteq \mathbb{R}^{N^d}$,

$$T_{d,N}(k_1, \dots, k_d) = T_{d,N}(k_{\sigma(1)}, \dots, k_{\sigma(d)})$$

- Example: $d = 2$





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- Basic question: how does the randomness of $T_{d,N}$ behave under repeated contractions?

- For $T_{d,N} \in \mathcal{S}_{d,N}$, $p \le d$, and vectors $v_1, \dots, v_p \in \mathbb{R}^N$, we define the contracted tensor

$$T_{d,N}[v_1 \otimes \cdots \otimes v_p] \in \mathcal{S}_{d-p,N}$$

by

$$T_{d,N}[v_1 \otimes \cdots \otimes v_p](k_1, \dots, k_{d-p})$$

$$= \sum_{\ell_1,\dots,\ell_p} T_{d,N}(k_1, \dots, k_{d-p}, \ell_1, \dots, \ell_p)\, v_1(\ell_1) \cdots v_p(\ell_p)$$

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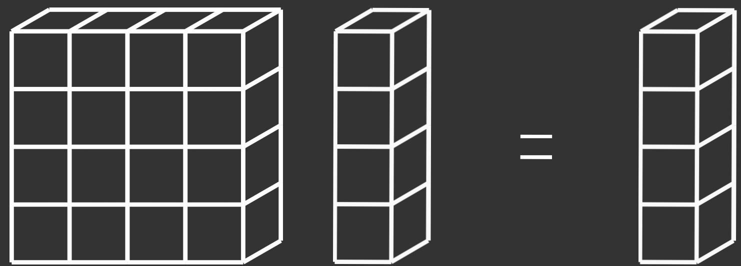

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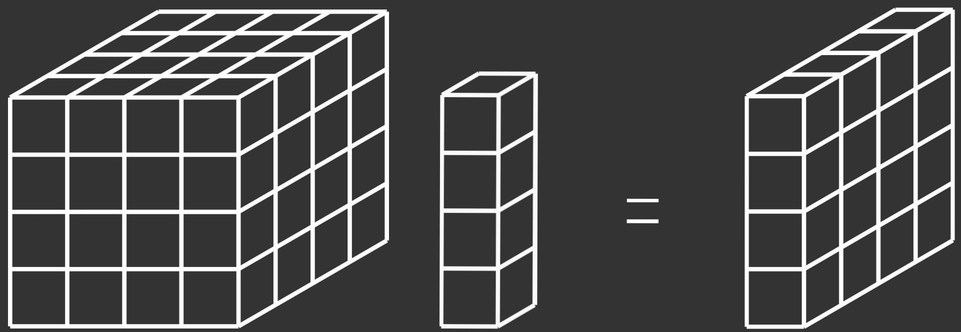

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Some observations:

- $T_{d,N} \in \mathcal{S}_{d,N}$ implies $T_{d,N}[v_1 \otimes \cdots \otimes v_p] \in \mathcal{S}_{d-p,N}$
- The order of the contracting vectors is immaterial
- The choice of contracted coordinates is also immaterial

- Basic question: how does the randomness of $T_{d,N}$ behave under repeated contractions?

- Example: $T_{1,N} \overset{d}{=} \mathcal{N}(\hat{0}, Id_N)$, $\quad v_N, w_N \in S^{N-1}$

$$\begin{pmatrix} T_{1,N}[v_N] \\ T_{1,N}[w_N] \end{pmatrix} \overset{d}{=} \mathcal{N}(\hat{0}, \mathcal{K}), \qquad \mathcal{K} = \begin{pmatrix} \langle v_N, v_N \rangle & \langle v_N, w_N \rangle \\ \langle w_N, v_N \rangle & \langle w_N, w_N \rangle \end{pmatrix}$$

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- Example: $T_{1,N} \overset{d}{=} (\underbrace{X_1, \dots, X_N}_{\text{i.i.d.},\ \mu=0,\ \sigma^2=1})$, $\quad \underbrace{v_N, w_N}_{\|\cdot\|_\infty = o(1)} \in S^{N-1}$

$$\begin{pmatrix} T_{1,N}[v_N] \\ T_{1,N}[w_N] \end{pmatrix} \xrightarrow{d} N(\hat{0}, \mathcal{K}), \quad \mathcal{K} = \lim_{N\to\infty} \begin{pmatrix} \langle v_N, v_N \rangle & \langle v_N, w_N \rangle \\ \langle w_N, v_N \rangle & \langle w_N, w_N \rangle \end{pmatrix}$$

- For our purposes, a Wigner matrix is a random real symmetric matrix $T_{2,N}$ such that:

  (1) the upper triangular entries are independent;

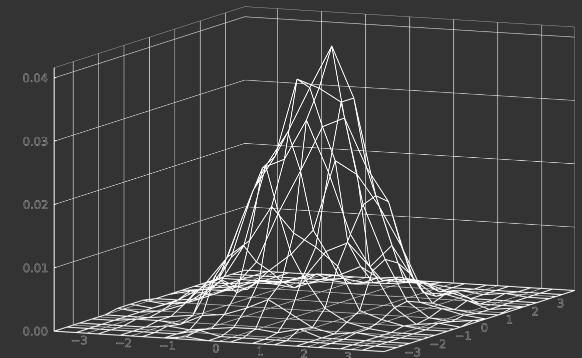
  (2) the off-diagonal entries are centered with variance $\frac{1}{2}$;

  (3) for any m,

$$\sup_{N \in \mathbb{N}} \sup_{k \leq \ell} \mathbb{E}\left[|T_{2,N}(k,\ell)|^m \right] = C_m < \infty$$

- Example: $T_{2,N}$ $\overbrace{\text{Wigner}}^{\sigma^2 = 1 \text{ on diagonal}}$, $\overbrace{v_N^{(1)}, v_N^{(2)}, w_N^{(1)}, w_N^{(2)}}^{\|\cdot\|_\infty = o(1)} \in S^{N-1}$

$$\begin{pmatrix} T_{2,N}\left[v_N^{(1)} \otimes v_N^{(2)}\right] \\ T_{2,N}\left[w_N^{(1)} \otimes w_N^{(2)}\right] \end{pmatrix} \xrightarrow{d} \mathcal{N}(\hat{0}, \mathcal{K}),$$



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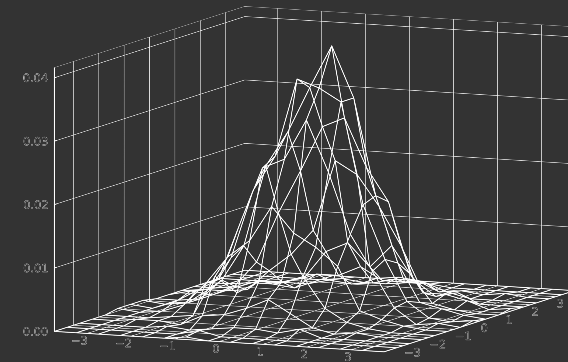

$$\mathcal{K} = \lim_{N \to \infty} \begin{pmatrix} \langle v_N^{(1)} \otimes v_N^{(2)}, v_N^{(1)} \otimes v_N^{(2)} \rangle & \langle v_N^{(1)} \otimes v_N^{(2)}, w_N^{(1)} \otimes w_N^{(2)} \rangle \\ \langle w_N^{(1)} \otimes w_N^{(2)}, v_N^{(1)} \otimes v_N^{(2)} \rangle & \langle w_N^{(1)} \otimes w_N^{(2)}, w_N^{(1)} \otimes w_N^{(2)} \rangle \end{pmatrix} ?$$

$$T_{d,N}[v_1 \otimes \cdots \otimes v_p](k_1, \dots, k_{d-p})$$

$$= \sum_{\ell_1, \dots, \ell_p} T_{d,N}(k_1, \dots, k_{d-p}, \ell_1, \dots, \ell_p)\, v_1(\ell_1) \cdots v_p(\ell_p)$$

Some observations:

- $T_{d,N} \in \mathcal{S}_{d,N}$ implies $T_{d,N}[v_1 \otimes \cdots \otimes v_p] \in \mathcal{S}_{d-p,N}$
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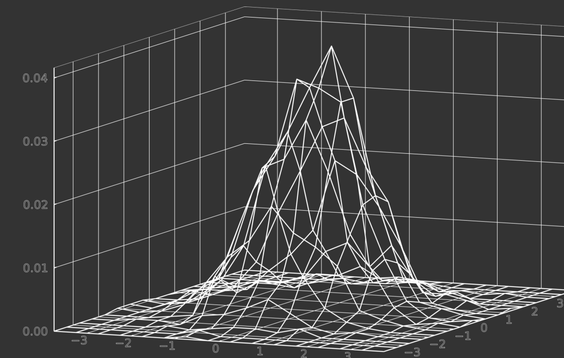
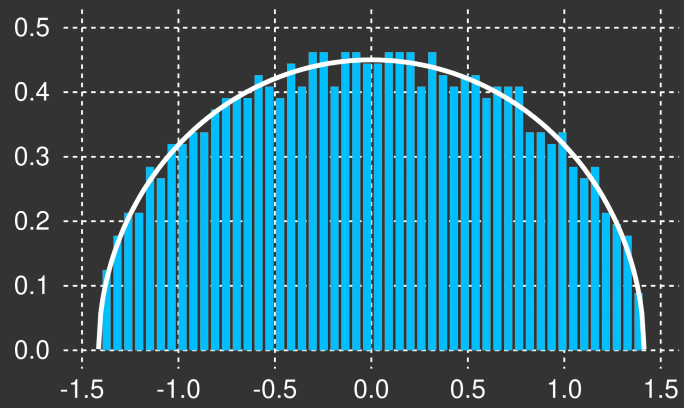


$$\mathcal{K} = \lim_{N \to \infty} \begin{pmatrix} \langle v_N^{(1)} \odot v_N^{(2)}, v_N^{(1)} \odot v_N^{(2)} \rangle & \langle v_N^{(1)} \odot v_N^{(2)}, w_N^{(1)} \odot w_N^{(2)} \rangle \\ \langle w_N^{(1)} \odot w_N^{(2)}, v_N^{(1)} \odot v_N^{(2)} \rangle & \langle w_N^{(1)} \odot w_N^{(2)}, w_N^{(1)} \odot w_N^{(2)} \rangle \end{pmatrix}$$

$$u_1 \odot \cdots \odot u_d = \frac{1}{d!} \sum_{\sigma \in S_d} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \in \mathcal{S}_{d,N}$$

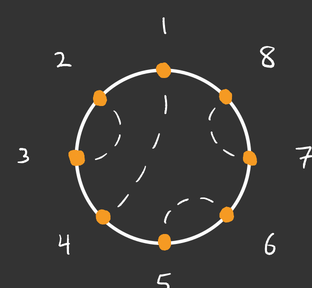
- (Wigner) The empirical spectral distribution of $W_N = \frac{1}{\sqrt{N}} T_{2,N}$ converges weakly almost surely to the semicircle distribution:

$$\mu(W_N) = \frac{1}{N} \sum_{k \in [N]} \delta_{\lambda_k(W_N)} \longrightarrow \frac{1}{\pi} (2 - x^2)_+^{\frac{1}{2}} \, dx$$



- (Voiculescu) Independent Wigner matrices $(W_N^{(i)})_{i \in I}$ are asymptotically free, converge in distribution to a multivariate semicircle $(s_i)_{i \in I} \overset{d}{=} SC(\hat{0}, \frac{1}{2} \mathrm{Id}_{\#(I)})$:

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \mathrm{Tr}\left[W_N^{(i_1)} \cdots W_N^{(i_m)} \right] \right] = \sum_{\pi \in \underbrace{NC_2}_{\text{non-crossing}}(m)} \prod_{\{j,k\} \in \pi} \underbrace{\frac{1}{2} \mathrm{Id}_{\#(I)}(i_j, i_k)}_{\mathcal{K}(i_j, i_k)}$$



- (Voiculescu) Independent Wigner matrices $(W_N^{(i)})_{i\in I}$ are asymptotically free, converge in distribution to a multivariate semicircle $(s_i)_{i\in I} \overset{d}{=} SC(\hat{O}, \frac{1}{2}\mathrm{Id}_{\#(I)})$:

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cf. $(X_i)_{i\in I} \overset{d}{=} N(\hat{O}, \mathcal{K})$:

$$\mathbb{E}[X_{i_1}\cdots X_{i_m}] = \sum_{\pi\in P_2(m)} \prod_{\{j,k\}\in\pi} \mathcal{K}(i_j, i_k)$$

- (Voiculescu) Independent Wigner matrices $(W_N^{(i)})_{i\in I}$ are asymptotically free, converge in distribution a.s. to a multivariate semicircle $(s_i)_{i\in I} \overset{d}{=} SC(\hat{0}, \frac{1}{2}\mathrm{Id}_{\#(I)})$:

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- Back to tensors: if $T_{d,N}$ is a random symmetric tensor, the corresponding contracted tensor ensemble (GCC21) is the family of random matrices $\left\{ T_{d,N}[u^{\otimes d-2}] \right\}_{u \in S^{N-1}}$
- What kind of randomness? A canonical distribution:

(GOE) $$\frac{1}{Z_N} e^{-\mathrm{Tr}(H^2)/2} dH$$

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- (GCC 21) For any sequence of unit vectors $u_N \in S^{N-1}$, the empirical spectral distribution of $W_N = \frac{1}{\sqrt{N}} T_{3,N}[u_N]$ converges weakly almost surely to the semicircle distribution with $\sigma^2 = \frac{1}{6}$

GOTE

- In general, W_N is not a Wigner matrix:

$$T_{3,N}[u_N](j,k) = \sum_{\ell} T_{3,N}(j,k,\ell)\, u_N(\ell)$$

- Proof relies on Stein's method and $d = 3$

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(GCC21)

Let us now rewrite the terms of (36) using these expressions (and also the fact that $\|v\| = 1$):

$$\begin{aligned}
\frac{1}{6}\sum_{m,\ell} v_\ell \mathbb{E}\left\{\frac{\partial r_{mj}}{\partial w_{im\ell}}\right\} = &-\frac{1}{\sqrt{N}}\frac{1}{6}\Bigg[\sum_m \mathbb{E}\{r_{mi}r_{jm}+r_{mm}r_{ji}\} - \mathbb{E}\{r_{ii}r_{ji}\} \\
&+\sum_{m,\ell} v_m v_\ell \mathbb{E}\{r_{mi}r_{j\ell}+r_{m\ell}r_{ji}\} - v_i\sum_m v_m \mathbb{E}\{r_{mi}r_{ji}\} \\
&+ v_i\sum_{m,\ell} v_\ell \mathbb{E}\{r_{mm}r_{j\ell}+r_{m\ell}r_{jm}\} - v_i\sum_m v_m \mathbb{E}\{r_{mm}r_{jm}\} \\
&-\sum_m v_m^2 \mathbb{E}\{r_{mi}r_{jm}+r_{mm}r_{ji}\} - v_i^2\sum_m \mathbb{E}\{r_{mi}r_{jm}+r_{mm}r_{ji}\} \\
&- v_i\sum_\ell v_\ell \mathbb{E}\{r_{ii}r_{j\ell}+r_{i\ell}r_{ji}\} + 4v_i^2\mathbb{E}\{r_{ii}r_{ji}\}\Bigg], \\
\frac{1}{6}v_i\sum_m \mathbb{E}\left\{\frac{\partial r_{mj}}{\partial w_{imi}}\right\} = &-\frac{1}{\sqrt{N}}\frac{1}{6}\Bigg[v_i^2\sum_m \mathbb{E}\{r_{mi}r_{jm}+r_{mm}r_{ji}\} + v_i\sum_m v_m \mathbb{E}\{r_{mi}r_{ji}\} \\
&- 2v_i^2\,\mathbb{E}\{r_{ii}r_{ji}\}\Bigg], \\
\frac{1}{6}\sum_\ell v_\ell \mathbb{E}\left\{\frac{\partial r_{ij}}{\partial w_{ii\ell}}\right\} = &-\frac{1}{\sqrt{N}}\frac{1}{6}\Bigg[v_i\sum_\ell v_\ell \mathbb{E}\{r_{ii}r_{j\ell}+r_{i\ell}r_{ji}\} + \mathbb{E}\{r_{ii}r_{ji}\} - 2v_i^2\,\mathbb{E}\{r_{ii}r_{ji}\}\Bigg], \\
\frac{1}{6}\sum_m v_m \mathbb{E}\left\{\frac{\partial r_{mj}}{\partial w_{imm}}\right\} = &-\frac{1}{\sqrt{N}}\frac{1}{6}\Bigg[\sum_m v_m^2 \mathbb{E}\{r_{mi}r_{jm}+r_{mm}r_{ji}\} + v_i\sum_m v_m \mathbb{E}\{r_{mm}r_{jm}\} \\
&- 2v_i^2\,\mathbb{E}\{r_{ii}r_{ji}\}\Bigg], \\
\frac{1}{3}v_i \mathbb{E}\left\{\frac{\partial r_{ij}}{\partial w_{iii}}\right\} = &-\frac{1}{\sqrt{N}}\frac{1}{3}v_i^2\,\mathbb{E}\{r_{ii}r_{ji}\}.
\end{aligned}$$

Combining these expressions, we obtain after some calculation

$$\begin{aligned}
\mathbb{E}\left\{\left[\tfrac{1}{\sqrt{N}}\mathbf{W}(v)R\right]_{ij}\right\} = &-\frac{1}{N}\Bigg[\frac{1}{6}\mathbb{E}\{[R^2]_{ij}\} + \frac{1}{6}\mathbb{E}\{r_{ij}\operatorname{tr}(R)\} + \frac{1}{6}\mathbb{E}\{[Rv]_i[Rv]_j\} + \frac{1}{6}\mathbb{E}\{v^\top Rv\, r_{ij}\} \\
&+\frac{1}{6}v_i\mathbb{E}\{\operatorname{tr}(R)[Rv]_j\} + \frac{1}{6}v_i\mathbb{E}\{[R^2v]_j\} - \frac{2}{3}v_i^2\mathbb{E}\{r_{ii}r_{ij}\}\Bigg].
\end{aligned}$$

Using now $R = -\frac{1}{z}I_N + \frac{1}{z}\frac{1}{\sqrt{N}}\mathbf{W}(v)R$ we get

$$\begin{aligned}
\mathbb{E}\{r_{ij}\} = &-\frac{1}{z}\delta_{ij} - \frac{1}{z}\frac{1}{N}\frac{1}{6}\mathbb{E}\{[R^2]_{ij}\} - \frac{1}{z}\frac{1}{N}\frac{1}{6}\mathbb{E}\{r_{ij}\operatorname{tr}(R)\} - \frac{1}{z}\frac{1}{N}\frac{1}{6}\mathbb{E}\{[Rv]_i[Rv]_j\} \\
&-\frac{1}{z}\frac{1}{N}\frac{1}{6}\mathbb{E}\{v^\top Rv\, r_{ij}\} - \frac{1}{z}\frac{1}{N}\frac{1}{6}v_i\mathbb{E}\{\operatorname{tr}(R)[Rv]_j\} - \frac{1}{z}\frac{1}{N}\frac{1}{6}v_i\mathbb{E}\{[R^2v]_j\} + \frac{1}{z}\frac{1}{N}\frac{2}{3}v_i^2\mathbb{E}\{r_{ii}r_{ij}\}.
\end{aligned}$$

Hence, summing over the diagonal of R

$$\begin{aligned}
m_N(z) := \mathbb{E}\left\{\frac{1}{N}\operatorname{tr}(R)\right\} = &-\frac{1}{z} - \frac{1}{z}\frac{1}{6}\frac{1}{N^2}\mathbb{E}\{\operatorname{tr}(R^2)\} - \frac{1}{z}\frac{1}{6}\frac{1}{N^2}\mathbb{E}\{\operatorname{tr}^2(R)\} - \frac{1}{z}\frac{1}{3}\frac{1}{N^2}\mathbb{E}\{v^\top R^2 v\} \\
&-\frac{1}{z}\frac{1}{3}\frac{1}{N^2}\mathbb{E}\{\operatorname{tr}(R)v^\top Rv\} + \frac{1}{z}\frac{2}{3}\frac{1}{N^2}\mathbb{E}\{v^\top \operatorname{Diag}^2(R)v\}.
\end{aligned} \tag{38}$$

- Question 1: what about higher order $d \geq 4$?

$$T_{3,N}[u_N](j,k) = \sum_{\ell} T_{3,N}(j,k,\ell)\, u_N(\ell)$$

$$T_{4,N}[u_N^{\otimes 2}](j,k) = \sum_{\ell_1,\ell_2} T_{4,N}(j,k,\ell_1,\ell_2)\, u_N(\ell_1)\, u_N(\ell_2)$$

- Question 2: universality for general tensor distributions?

- Question 3: general contractions $u_N^{(1)} \otimes \cdots \otimes u_N^{(d-2)} \neq u_N^{\otimes d-2}$?

- Question 4: joint behavior of $\left\{ T_{d,N}[u_N^{\otimes d-2}] \right\}_{u_N \in S^{N-1}}$?

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- For our purposes, a Wigner tensor is a random real symmetric tensor $T_{d,N}$ such that:

  (1) the upper triangular entries are independent;

  (2) the off-diagonal entries are centered;

  (2′) entries with $\#(k_1, \dots, k_d) \geq 3$ have variance $\binom{d}{b_1, \dots, b_N}^{-1}$;

  (3) for any m,

$$\sup_{N \in \mathbb{N}} \sup_{\vec{k}} \mathbb{E}\left[|T_{d,N}(k_1, \dots, k_d)|^m \right] = C_m < \infty$$

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(3) for any m, $\boxed{\frac{1}{Z_{d,N}} e^{-\|H\|_F^2/2}\, dH} \longrightarrow \binom{d}{b_1, \dots, b_N}^{-1}$;

$$\sup_{N \in \mathbb{N}} \sup_{\vec{k}} \mathbb{E}\left[|T_{d,N}(k_1, \dots, k_d)|^m \right] = C_m < \infty$$

- For vectors $u_1, \dots, u_d \in \mathbb{R}^N$, define the symmetrization

$$u_1 \odot \cdots \odot u_d = \frac{1}{d!} \sum_{\sigma \in S_d} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \in \mathcal{S}_{d-p,N}.$$

- For any sequence of families of unit vectors $\{u_N^{(i,j)}\}_{i \in I,\, j \in [d-2]}$, let $\mathcal{K}^{(N)} = \left(\mathcal{K}^{(N)}(i,i')\right)_{i,i' \in I}$ be the rescaled Gram matrix of the symmetrizations:

$$\mathcal{K}^{(N)}(i,i') = \frac{1}{d(d-1)} \left\langle u_N^{(i,1)} \odot \cdots \odot u_N^{(i,d)},\ u_N^{(i',1)} \odot \cdots \odot u_N^{(i',d)} \right\rangle.$$

- (AGV21) Let $T_{d,N}$ be a Wigner tensor and define $(W_N^{(i)})_{i\in I} = \left(\frac{1}{\sqrt{N}} T_{d,N}[u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)}]\right)_{i\in I}$. Then $(W_N^{(i)})_{i\in I}$ converges in distribution a.s. iff the limits

$$\mathcal{K}(i,i') = \lim_{N\to\infty} \mathcal{K}^{(N)}(i,i')$$

exist, in which case $(W_N^{(i)})_{i\in I} \to SC(\hat{O}, \mathcal{K})$.

- $\left(W_N^{(i)}\right)_{i\in I} = \left(\frac{1}{\sqrt{N}} T_{d,N}\left[u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)}\right]\right)_{i\in I}$

$$\mathcal{K}(i,i') = \lim_{N\to\infty} \frac{1}{d(d-1)} \left\langle u_N^{(i,1)} \odot \cdots \odot u_N^{(i,d)},\ u_N^{(i',1)} \odot \cdots \odot u_N^{(i',d)} \right\rangle$$

- $\left(W_N^{(i)}\right)_{i\in I} \to SC(\hat{O}, \mathcal{K})$ in distribution a.s.

- What about a single matrix $W_N^{(i)}$?

- First, assume $u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)} = u_N^{\otimes d-2}$. Then

$$\mathcal{K}(i,i) = \frac{1}{d(d-1)}$$

- $\left(W_N^{(i)}\right)_{i\in I} = \left(\frac{1}{\sqrt{N}} T_{d,N}\left[u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)}\right]\right)_{i\in I}$

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$$\left\langle u_N^{(i,1)} \odot \cdots \odot u_N^{(i,d)}, \; u_N^{(i',1)} \odot \cdots \odot u_N^{(i',d)} \right\rangle = \frac{1}{(d-2)!} \operatorname{per}\left[\left(\left\langle u_N^{(i,j)}, \; u_N^{(i',k)} \right\rangle\right)_{j,k\in[d-2]}\right]$$

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≥ 1 (Marcus)

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$$\mathbb{P}\left[\max_{\substack{m \le m_0 \\ i_1,\dots,i_m \in I_0}} \left| \frac{1}{N} \mathrm{Tr}\left[W_N^{(i_1)} \cdots W_N^{(i_m)}\right] - \sum_{\pi \in NC_2(m)} \prod_{\{j,k\}\in\pi} \mathcal{K}_N(i_j, i_k) \right| > \varepsilon\right] < \frac{C}{N^M}$$

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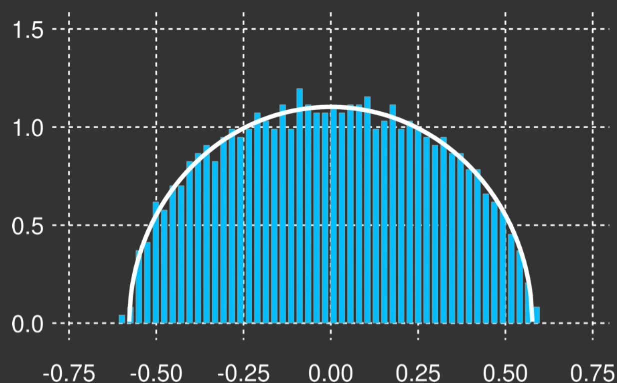

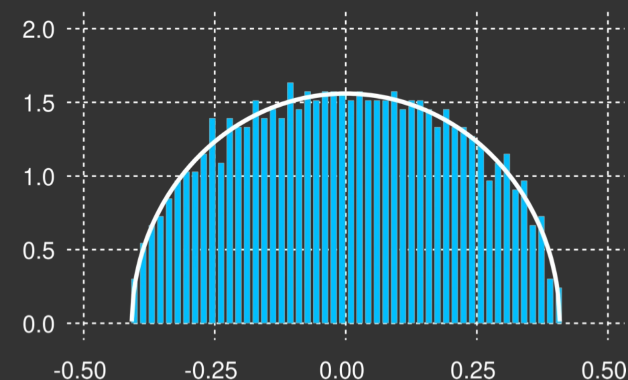

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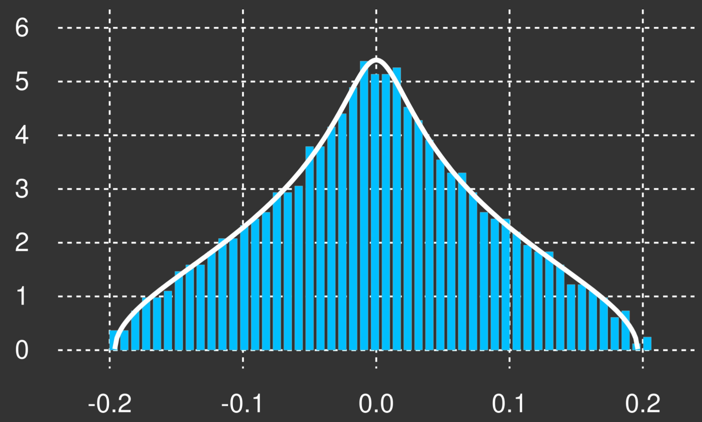

- Is this covered by some result for dependent random matrices? Thankfully, no.
- (SSB05) Partitioned entries (but constrained block size)
- (GNT15) Conditional centeredness ($\mathbb{E}[X_{i,j} \mid \mathcal{F}_{i,j}] = 0$)
- (BMP15) Array representation ($X_{i,j} = g(Y_{i-k,\ell-j} : (k,\ell) \in \mathbb{Z}^2)$)
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- (Penrose) Tensors represented as shapes such as boxes and triangles; indices notated by lines

- (RMT) Matrices represented by lines; indices by circles

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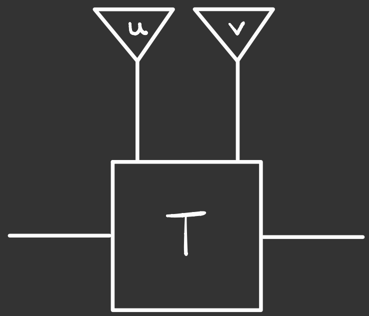

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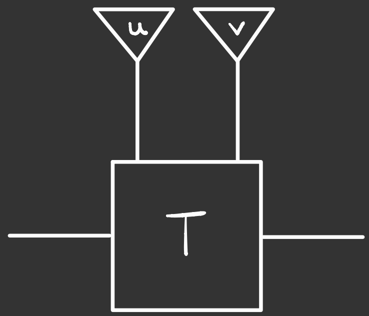

$T[u \otimes v]$

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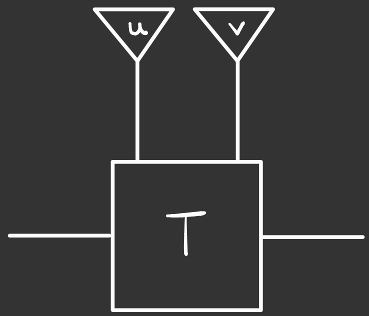

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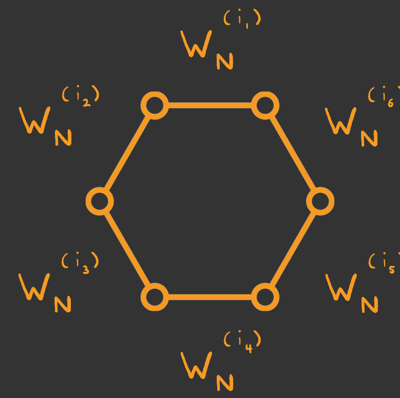

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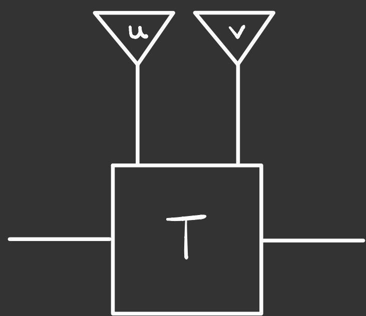
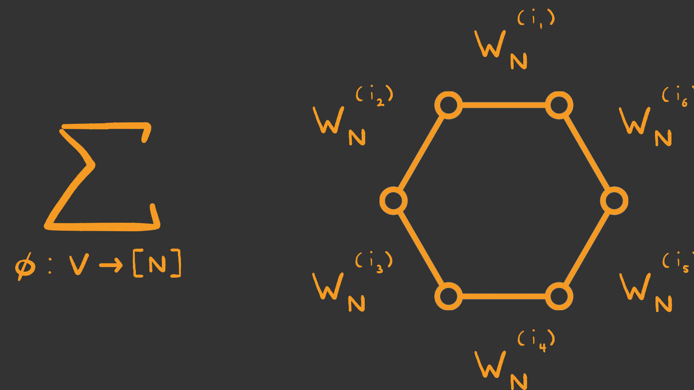


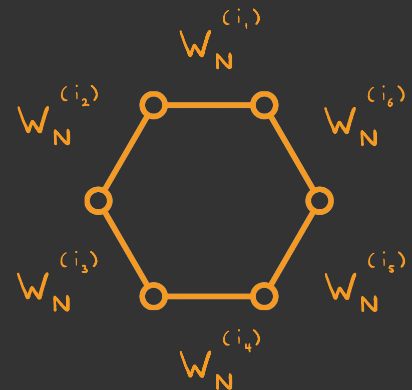


$T[u \otimes v]$

$\mathrm{Tr}[W_N^{(i_1)} \cdots W_N^{(i_6)}]$

- (RMT) Matrices represented by lines; indices by circles

$$\sum_{\phi : V \to [N]}$$



$$\mathrm{Tr}\left[W_N^{(i_1)} \cdots W_N^{(i_6)}\right]$$

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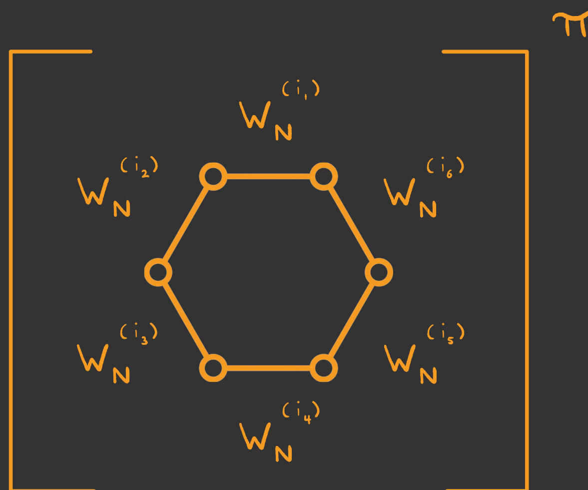

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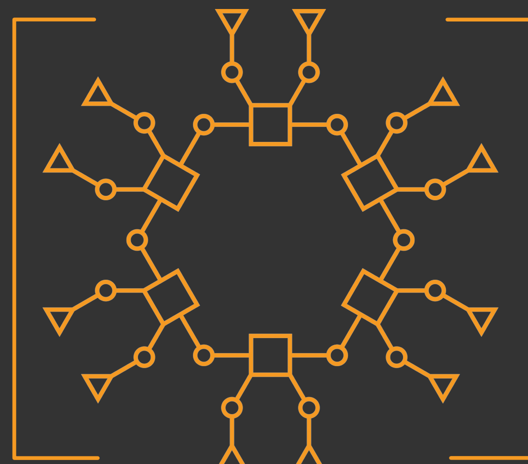

$$^{\pi}$$

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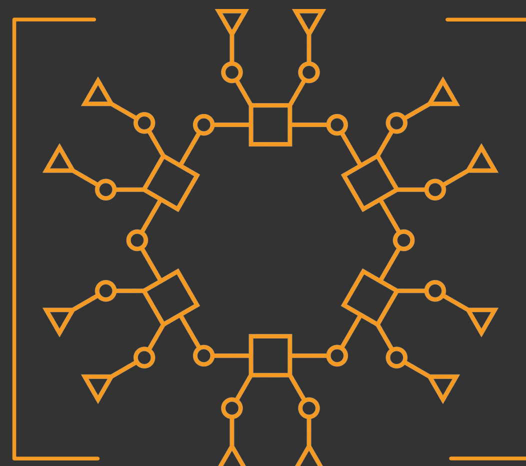

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- What is the complication?

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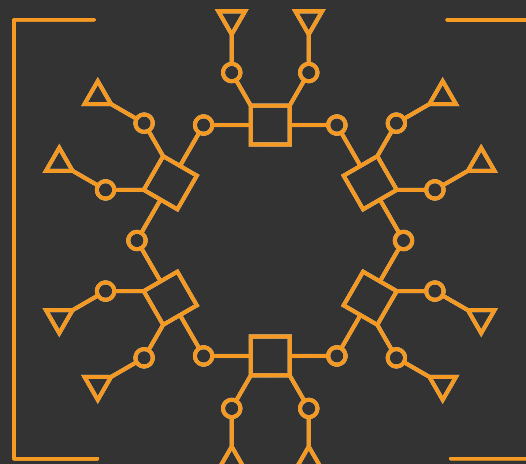

π

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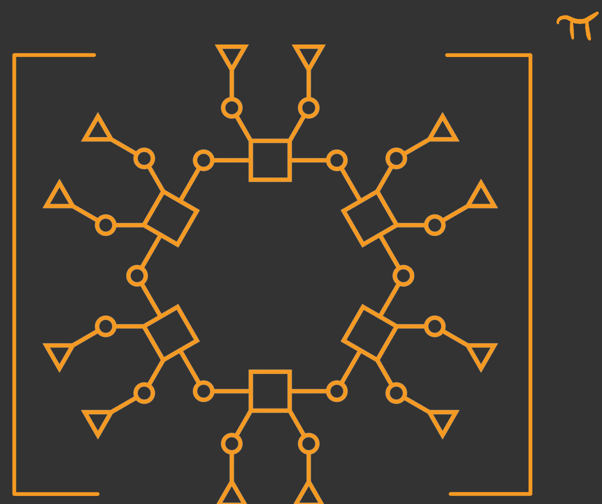

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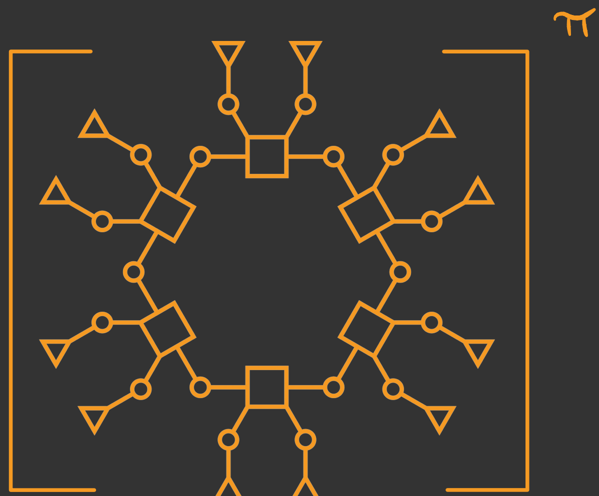

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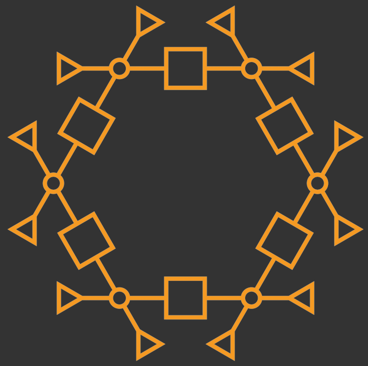

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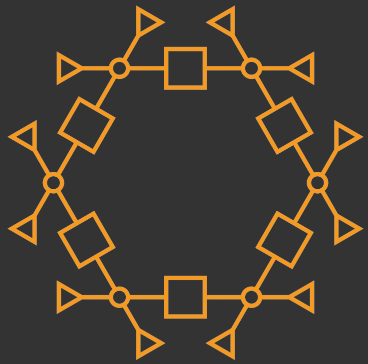
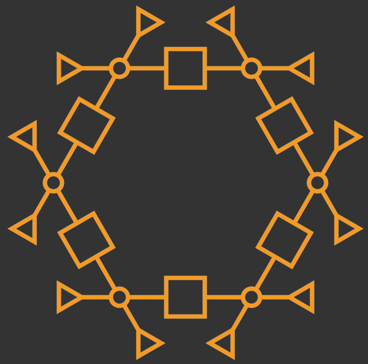


- Naive bound: $O(N^2)$

- What is the complication? Too many vertices!
- What is the solution? Unit vectors!



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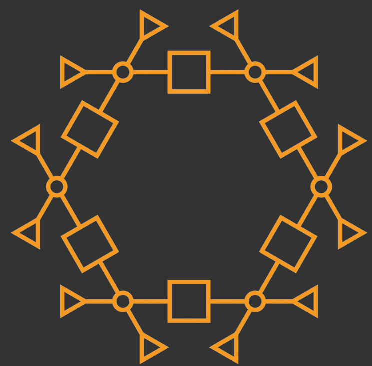
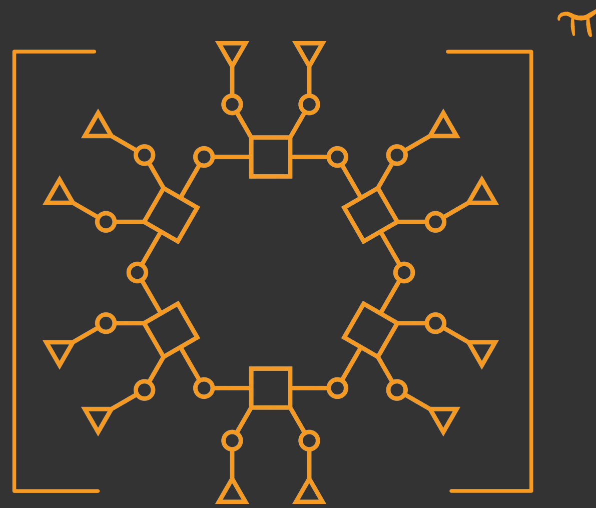


- Naive bound: $O(N^2)$ / True bound: $O(N^{-4})$

$$\sum_{\pi \in P(V)} \; \sum_{\substack{\phi : V^{\pi} \to [N] \\ \text{injective}}}$$



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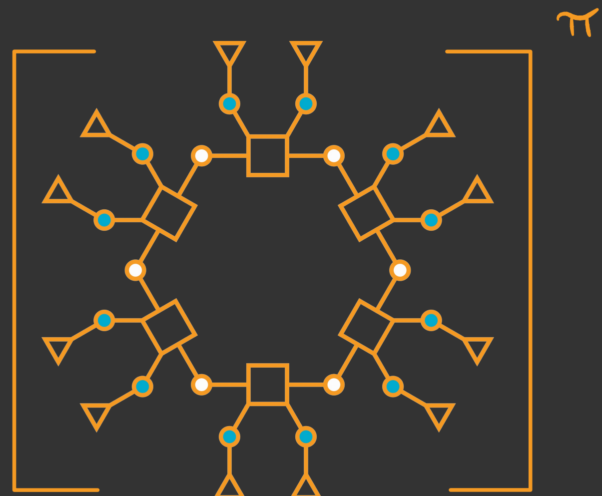
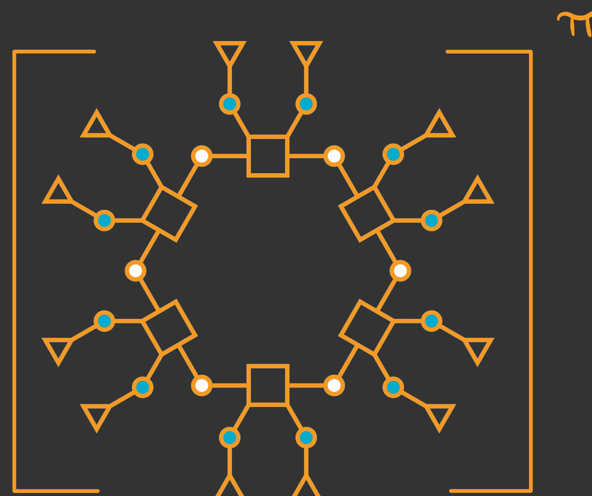


inner/outer

$$\sum_{\pi \in P(V)} \sum_{\substack{\phi : V^{\pi} \to [N] \\ \text{injective}}}$$

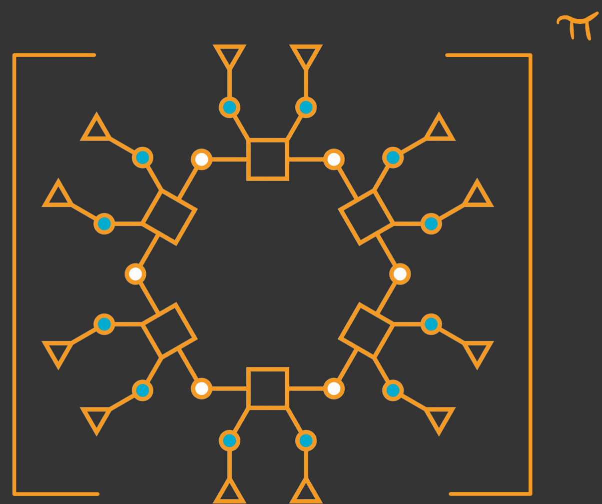


inner/outer

$$\sum_{\pi \in P(V)} \sum_{\phi : V^{\pi} \to [N]}$$

s.t. $\pi = \pi|_{V_{\text{inner}}} \cup \pi|_{V_{\text{outer}}}$

injective



inner/outer

π

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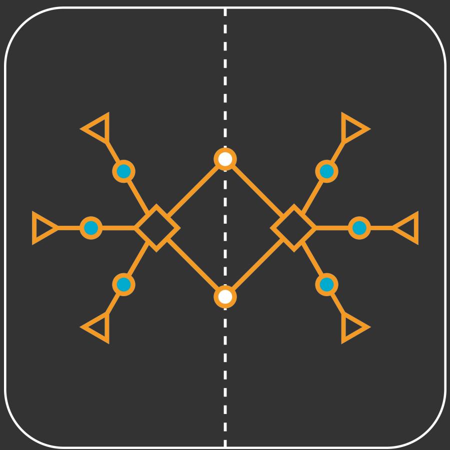

inner/outer

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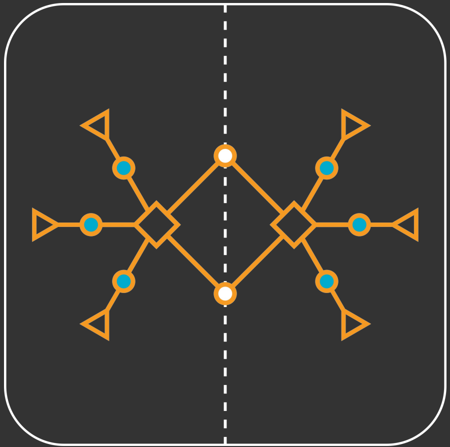

(2′) entries with $\#(k_1, \ldots, k_d) \geq 3$ have variance

$$\binom{d}{b_1, \ldots, b_N}^{-1};$$

- Consider two disjoint copies $\overset{\text{left}}{[n]_L}$ and $\overset{\text{right}}{[n]_R}$ of the set $[n] = \{1, \dots, n\}$. A uniform block permutation of $[n]$ is a partition $\pi \in P([n]_L \sqcup [n]_R)$ such that each block $B \in \pi$ has the same number of left and right elements:

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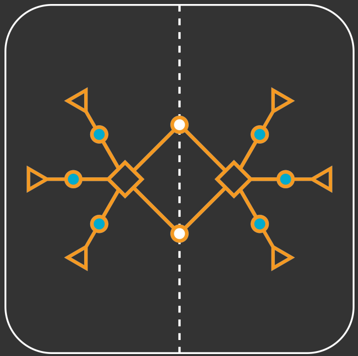
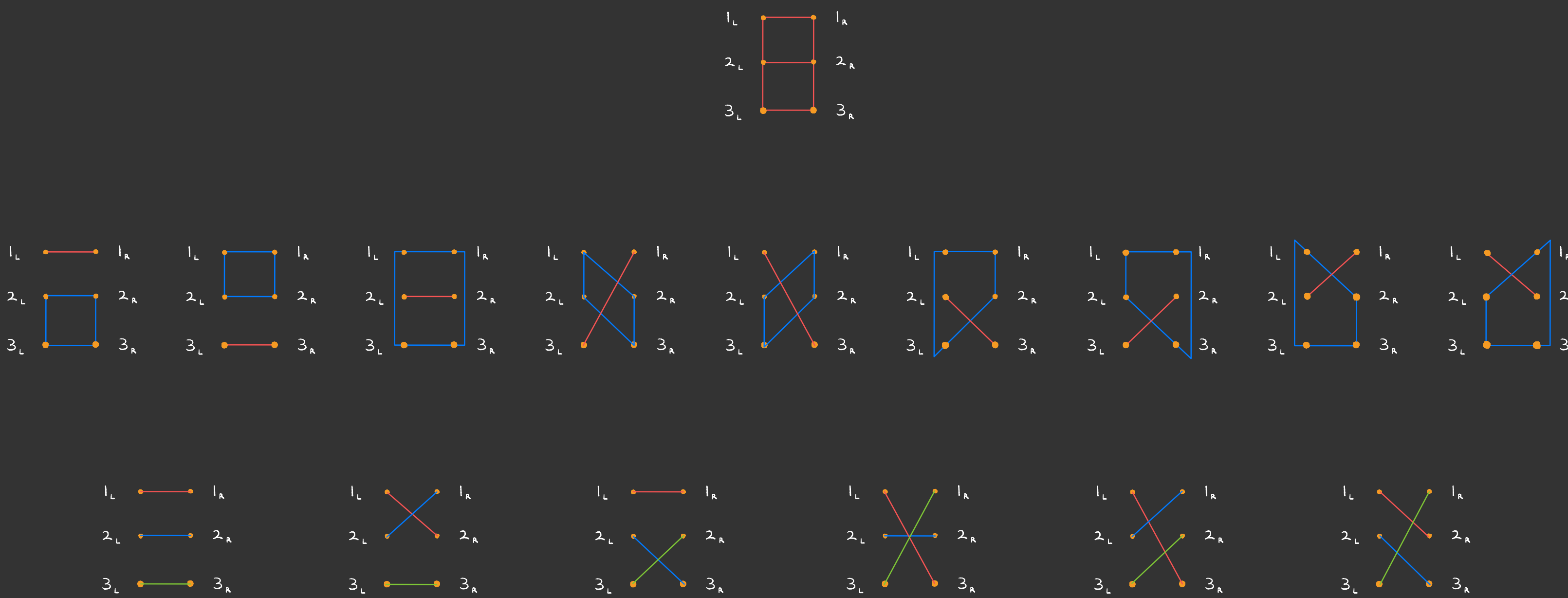
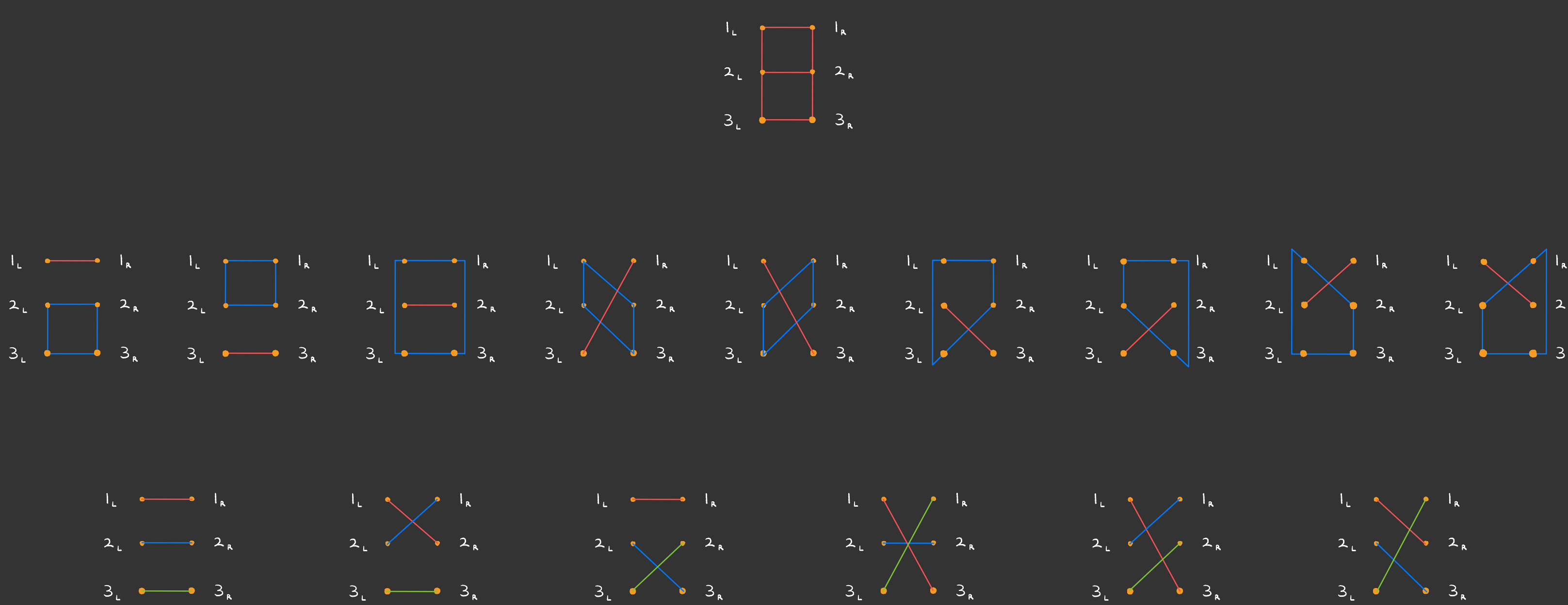




UBP(3)



$(UBP(3), \leq)$

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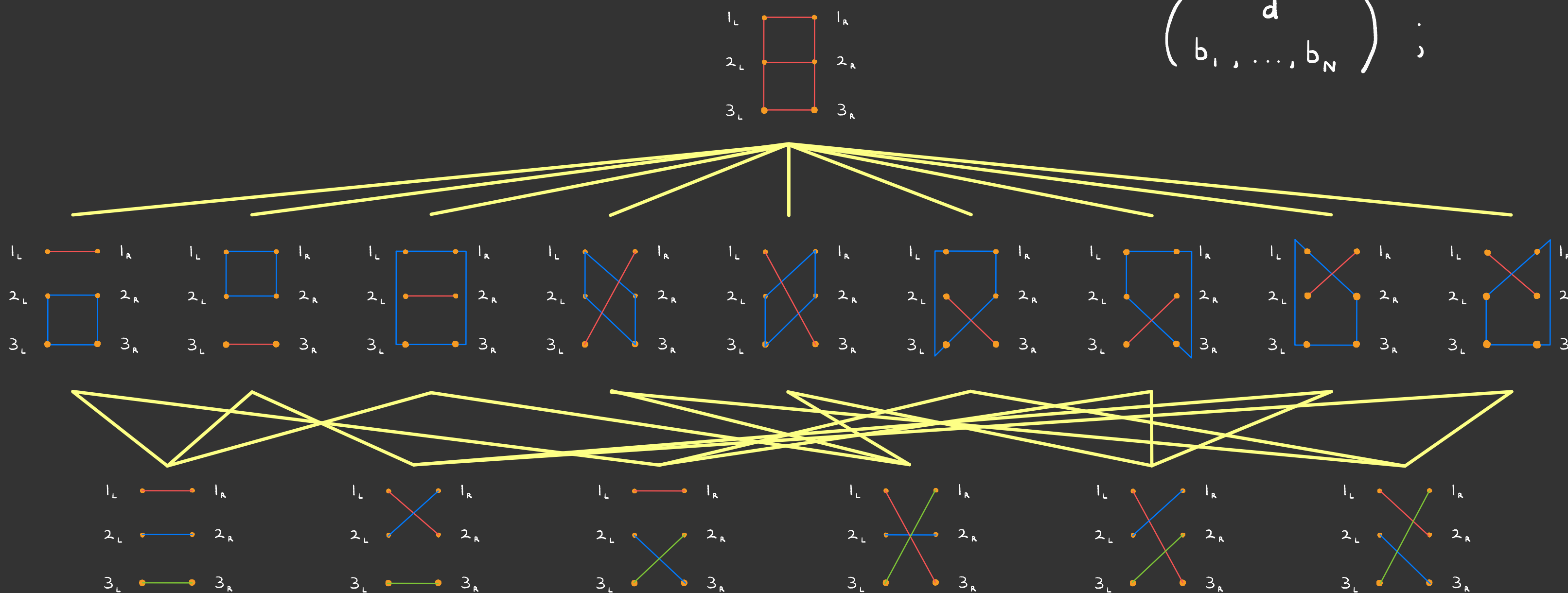

$(\mathrm{UBP}(3), \leq)$

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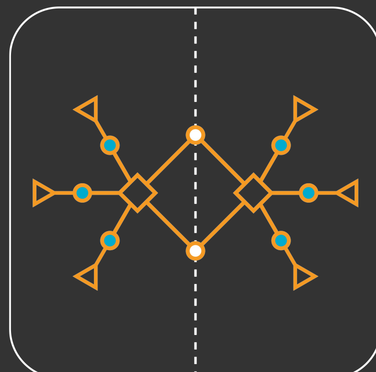
inner/outer

π

$$\sum_{\substack{\pi \in P(V) \\ \text{s.t.} \\ \pi = \pi|_{V_{\text{inner}}} \cup \pi|_{V_{\text{outer}}}}} \ \sum_{\substack{\phi : V^{\pi} \to [N] \\ \text{injective}}}$$



$$= \frac{1}{d!} \sum_{\sigma \in S_{d-2}} \prod_{j=1}^{d-2} \left\langle u_N^{(i,j)}, u_N^{(i',\sigma(j))} \right\rangle$$

(2′) entries with $\#(k_1, \dots, k_d) \geq 3$ have variance $\binom{d}{b_1, \dots, b_N}^{-1}$;