Bias in the Free World

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Joint work with L. Goldstein (USC)

Random sampling nearly always incurs some form of bias. Transforms to compensate for bias are often connected to Stein's method, and have found applications to normal approximation, waiting-time paradoxes, tightness, Skorohod embedding, concentration of measure, infinite divisibility, and many other far-flung ideas.

In 1997, Goldstein and Reinert introduced the **zero bias** transform for centered distributions. Given (the law of) a centered random variable X of variance σ^2, its zero bias X^* is (the law) determined by the functional equation

$$\mathbb{E}[Xf(X)] = \sigma^2\,\mathbb{E}[f'(X^*)] \quad \text{for all } \mathrm{Lip}_1 \text{ functions } f.$$

The map $X \mapsto X^*$ has good smoothing/regularity properties, and interacts well with independent sums: if $X_1, \ldots, X_n$ are i.i.d. then

$$(X_1 + X_2 + \cdots + X_n)^* \stackrel{d}{=} X_1^* + X_2 + \cdots + X_n.$$

This provides for elegant and sharp proofs of Berry–Esseen theorems. More recently, in joint work with U. Schmock, they found an unexpected connection to infinite divisibility: a centered L^2 random variable X is infinitely divisible if and only if

$$X^* \stackrel{d}{=} X + UY$$

where X, U, Y are independent and U is uniform on $[0,1]$. The proof is probabilistic and connects the above property directly to the Kévy–Khintchine formula: the law of Y is the associated Lévy–Khintchine measure.

I will explain these ideas in this talk, as well as parallels Goldstein and I have discovered in *free probability*, where the classical notion of independence is replaced by *free independence* modeled on freeness in group theory. We introduce another transform, the **free zero bias** X°, satisfying the functional equation

$$\mathbb{E}[Xf(X)] = \sigma^2\,\mathbb{E}[\partial f(X^\circ)] \quad \text{for all } \mathrm{Lip}_1 \text{ functions } f$$

where ∂f is the *free difference quotient*, a noncommutative derivative arising from functional calculus and perturbation theory of eigenvalues. I will describe our results providing precise (but intriguingly different) analogs of the

properties and applications of the classical zero bias, concluding with a new and more probabilistic approach to free infinite divisibility. In particular, we prove that every probability measure is a free Lévy–Khintchine measure.