Efficient Optimal Strategies for Universal Prediction

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Joint work with Yasin Abbasi-Yadkori, Wouter Koolen, Alan Malek, Eiji Takimoto, Manfred Warmuth.
Online Prediction

A repeated game:

At round $t$:

1. Player chooses prediction $a_t \in A$.
2. Adversary chooses outcome $y_t \in Y$.
3. Player incurs loss $\ell(a_t, y_t)$.

Player's aim:

$$\sum_{t=1}^{T} \ell(a_t, y_t) - \inf \sum_{t=1}^{T} \ell(a, y_t).$$

$$\ell(a_t, y_t) = \|a_t - y_t\|_2.$$
Online Prediction

A repeated game:
At round $t$:

1. Player chooses prediction $a_t \in A$. 

---

$\ell(a_t, y_t) = \|a_t - y_t\|_2$. 

Player's aim: 
$\frac{1}{T} \sum_{t=1}^{T} \ell(a_t, y_t) - \inf \frac{1}{T} \sum_{t=1}^{T} \ell(\cdot, y_t)$. 

...
Online Prediction

A repeated game:

At round $t$:
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Player's aim:
$$
\sum_{t=1}^{T} \ell(a_t, y_t) - \inf \sum_{t=1}^{T} \ell(\cdot, y_t).
$$

$$
\ell(a_t, y_t) = ||a_t - y_t||_2.
$$
Online Prediction

A repeated game:

At round $t$:

1. Player chooses prediction $a_t \in \mathcal{A}$.
2. Adversary chooses outcome $y_t \in \mathcal{Y}$.
3. Player incurs loss $\ell(a_t, y_t)$.

\[ \ell(a_t, y_t) = \|a_t - y_t\|^2. \]
A repeated game:

At round $t$:

1. Player chooses prediction $a_t \in A$.
2. Adversary chooses outcome $y_t \in Y$.
3. Player incurs loss $\ell(a_t, y_t)$.

Player's aim:

$$\sum_{t=1}^{T} \ell(a_t, y_t) - \inf_{\sum_{t=1}^{T} \ell(a_t, y_t)}$$

$$\ell(a_t, y_t) = \|a_t - y_t\|_2$$
Online Prediction

A repeated game:

At round $t$:

1. Player chooses prediction $a_t \in A$.
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Player's aim:

$$\sum_{t=1}^{T} \ell(a_t, y_t) - \inf \sum_{t=1}^{T} \ell(a_t, y_t)$$
Online Prediction

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Player's aim:

$$\sum_{t=1}^{T} \ell(a_t, y_t) - \inf \sum_{t=1}^{T} \ell(a_t, y_t).$$
Online Prediction

A repeated game:

At round \( t \):

1. Player chooses prediction \( a_t \in \mathcal{A} \).
2. Adversary chooses outcome \( y_t \in \mathcal{Y} \).
3. Player incurs loss \( \ell(a_t, y_t) \).

\[ \ell(a_t, y_t) = \| a_t - y_t \|^2 \]
Online Prediction

A repeated game:

At round $t$:

1. Player chooses prediction $a_t \in A$.
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$\ell(a_t, y_t) = \|a_t - y_t\|_2$. 
Online Prediction

A repeated game:

At round $t$:
1. Player chooses prediction $a_t \in \mathcal{A}$.
2. Adversary chooses outcome $y_t \in \mathcal{Y}$.
3. Player incurs loss $\ell(a_t, y_t)$.

Player’s aim:
Minimize regret:

$$
\sum_{t=1}^{T} \ell(a_t, y_t) - \inf_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t).
$$
A repeated game:

At round $t$:
1. Player chooses prediction $a_t \in A$.
2. Adversary chooses outcome $y_t \in Y$.
3. Player incurs loss $\ell(a_t, y_t)$.

Player’s aim:
Minimize regret wrt comparison $C$:

$$\sum_{t=1}^{T} \ell(a_t, y_t) - \inf_{\hat{a} \in C} \sum_{t=1}^{T} \ell(\hat{a}_t, y_t)$$
Universal prediction:
very weak assumptions on process generating the data.
Online Prediction Games: Why

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Online Prediction Games: Why

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- Deterministic heart of a decision problem.
- Gives robust statistical methods.
- Typically streaming, so very scalable.
- This talk: Minimax optimal strategies.
### Online Prediction Games

#### Regret

\[
V_T(Y, A) = \min_{a_1 \in A} \max_{y_1 \in Y} \cdots \min_{a_T \in A} \max_{y_T \in Y} \left( \sum_{t=1}^T \ell(a_t, y_t) - \min_{a \in A} \sum_{t=1}^T \ell(a, y_t) \right)
\]
### Online Prediction Games

#### Minimax Regret

\[
\begin{align*}
    V_T(Y, A) &= \min_{a_1 \in A} \max_{y_1 \in Y} \cdots \min_{a_T \in A} \max_{y_T \in Y} \left( T \sum_{t=1}^T \ell(a_t, y_t) - \min_{a \in A} \sum_{t=1}^T \ell(a, y_t) \right) \\
    &= \max_{y_T \in Y} \left( T \sum_{t=1}^T \ell(S^*(y_t-1), y_t) - \min_{a \in A} \sum_{t=1}^T \ell(a, y_t) \right)
\end{align*}
\]
Online Prediction Games

Minimax Regret

\[
\min_{a_1 \in A} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in A} \sum_{t=1}^{T} \ell(a, y_t) \right)
\]
Online Prediction Games

Minimax Regret

\[
\min_{a_1 \in A} \max_{y_1 \in Y} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in A} \sum_{t=1}^{T} \ell(a, y_t) \right)
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Online Prediction Games

Minimax Regret

\[
\min_{a_1 \in A} \max_{y_1 \in Y} \cdots \min_{a_T \in A} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in A} \sum_{t=1}^{T} \ell(a, y_t) \right)
\]
Online Prediction Games

Minimax Regret

$$\min_{a_1 \in \mathcal{A}} \max_{y_1 \in \mathcal{Y}} \cdots \min_{a_T \in \mathcal{A}} \max_{y_T \in \mathcal{Y}} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t) \right)$$
Online Prediction Games

The value of the game: Minimax Regret

\[ V_T(\mathcal{Y}, \mathcal{A}) = \min_{a_1 \in \mathcal{A}} \max_{y_1 \in \mathcal{Y}} \cdots \min_{a_T \in \mathcal{A}} \max_{y_T \in \mathcal{Y}} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t) \right) \]
The value of the game: Minimax Regret

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Strategy:

\[ S : \bigcup_{t=0}^{T} \mathcal{Y}^t \rightarrow \mathcal{A}. \]
Online Prediction Games

The value of the game: Minimax Regret

\[ V_T(\mathcal{Y}, \mathcal{A}) = \min_{a_1 \in \mathcal{A}} \max_{y_1 \in \mathcal{Y}} \cdots \min_{a_T \in \mathcal{A}} \max_{y_T \in \mathcal{Y}} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t) \right) \]

Strategy:

\[ S : \bigcup_{t=0}^{T} \mathcal{Y}^t \rightarrow \mathcal{A}. \]

\[ V_T(\mathcal{Y}, \mathcal{A}) = \min_S \max_{y_1^T \in \mathcal{Y}^T} \left( \sum_{t=1}^{T} \ell(S(y_1^{t-1}), y_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t) \right) \]
Online Prediction Games

The value of the game: Minimax Regret

\[
V_T(\mathcal{Y}, \mathcal{A}) = \min_{a_1 \in \mathcal{A}} \max_{y_1 \in \mathcal{Y}} \cdots \min_{a_T \in \mathcal{A}} \max_{y_T \in \mathcal{Y}} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t) \right)
\]

Minimax Optimal Strategy:

\[
S^* : \bigcup_{t=0}^{T} \mathcal{Y}^t \to \mathcal{A}.
\]

\[
V_T(\mathcal{Y}, \mathcal{A}) = \min_S \max_{y_1^T \in \mathcal{Y}^T} \left( \sum_{t=1}^{T} \ell(S(y_1^{t-1}), y_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t) \right) = \max_{y_1^T \in \mathcal{Y}^T} \left( \sum_{t=1}^{T} \ell(S^*(y_1^{t-1}), y_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t) \right).
\]
Online Prediction Games

Questions

Minimax regret?

Optimal player's strategy?

Efficiently computable?

Optimal adversary's strategy?

How do they depend on $\ell$ loss, $\ell(a, y)$:

1. $\|a - y\|_2^2, a, y \in \mathbb{R}^d$.

2. $(x^\top a - y)^2$.

3. $- \log a(y)$, $a \in \{p_\theta : \theta \in \Theta\}$. 
Online Prediction Games

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Online Prediction Games

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Online Prediction Games

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\[ \|a - y\|_2^2, \quad a, y \in \mathbb{R}^d. \]

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Online Prediction Games

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Online Prediction Games

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- How do they depend on $\ell$?
Online Prediction Games

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loss, $\ell(a, y)$:
- $1 \parallel a - y \parallel_2^2$
- $a, y \in \mathbb{R}^d$.
Online Prediction Games

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Online Prediction Games

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Online Prediction Games

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- Minimax regret?
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- How do they depend on $\ell, \mathcal{Y}, \mathcal{A}$?

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Online Prediction Games

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Online Prediction Games

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Outline

Computing minimax optimal strategies.


Part 2: Linear regression.

Time series forecasting.
Computing minimax optimal strategies.
Computing minimax optimal strategies.

Outline

- Computing minimax optimal strategies.
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• Computing minimax optimal strategies.
• Part 1: Euclidean loss.
• Part 2: Linear regression.
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Computing minimax optimal strategies.


Part 2: Linear regression.

Time series forecasting.
Computing minimax optimal strategies

The value of the game:

\[ V_T(Y, A) = \min_{a_1 \in A} \max_{y_1 \in Y} \ldots \min_{a_T \in A} \max_{y_T \in Y} \left( \sum_{t=1}^T \ell(a_t, y_t) - \min_{a \in A} \sum_{t=1}^T \ell(a, y_t) \right). \]

Recursion for the value-to-go, given a history:

\[ V(y_1, \ldots, y_T) := -\min_{a_T} \sum_{t=1}^T \ell(a_t, y_t), \quad V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)) \]

\[ V_T(Y, A) = V(), \quad S^* = \arg \min_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)) \]
Computing minimax optimal strategies

The value of the game:

\[ V_T(Y, A) = \min_{a_1 \in A} \max_{y_1 \in Y} \ldots \min_{a_T \in A} \max_{y_T \in Y} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in A} \sum_{t=1}^{T} \ell(a, y_t) \right). \]

Recursion for the value-to-go, given a history:

\[ V_T(y_1, \ldots, y_{t-1}) = \min_{a_t \in A} \max_{y_t \in Y} \left( \ell(a_t, y_t) + V_T(y_1, \ldots, y_{t-1}) \right). \]
Computing minimax optimal strategies

The value of the game:

\[ V_T(\mathcal{Y}, \mathcal{A}) = \min_{a_1 \in \mathcal{A}} \max_{y_1 \in \mathcal{Y}} \cdots \min_{a_T \in \mathcal{A}} \max_{y_T \in \mathcal{Y}} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t) \right). \]

Recursion for the value-to-go, given a history:

\[ V(y_1, \ldots, y_T) := - \min_{a} \sum_{t=1}^{T} \ell(a, y_t), \]
Computing minimax optimal strategies

The value of the game:

\[ V_T(\mathcal{Y}, \mathcal{A}) = \min_{a_1 \in \mathcal{A}} \max_{y_1 \in \mathcal{Y}} \cdots \min_{a_T \in \mathcal{A}} \max_{y_T \in \mathcal{Y}} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t) \right) . \]

Recursion for the value-to-go, given a history:

\[ V(y_1, \ldots, y_T) := - \min_a \sum_{t=1}^{T} \ell(a, y_t), \]

\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} \left( \ell(a_t, y_t) + V(y_1, \ldots, y_t) \right) , \]
Computing minimax optimal strategies

The value of the game:

\[ V_T(\mathcal{Y}, \mathcal{A}) = \min_{a_1 \in \mathcal{A}} \max_{y_1 \in \mathcal{Y}} \cdots \min_{a_T \in \mathcal{A}} \max_{y_T \in \mathcal{Y}} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t) \right). \]

Recursion for the value-to-go, given a history:

\[
\begin{align*}
V(y_1, \ldots, y_T) &:= - \min_{a} \sum_{t=1}^{T} \ell(a, y_t), \\
V(y_1, \ldots, y_{t-1}) &:= \min_{a_t} \max_{y_t} \left( \ell(a_t, y_t) + V(y_1, \ldots, y_t) \right), \\
V_T(\mathcal{Y}, \mathcal{A}) &= V().
\end{align*}
\]
Computing minimax optimal strategies

The value of the game:

\[ V_T(\mathcal{Y}, \mathcal{A}) = \min_{a_1 \in \mathcal{A}} \max_{y_1 \in \mathcal{Y}} \cdots \min_{a_T \in \mathcal{A}} \max_{y_T \in \mathcal{Y}} \left( \sum_{t=1}^{T} \ell(a_t, y_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t) \right). \]

Recursion for the value-to-go, given a history:

\[ V(y_1, \ldots, y_T) := -\min_a \sum_{t=1}^{T} \ell(a, y_t), \]

\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)), \]

\[ V_T(\mathcal{Y}, \mathcal{A}) = V(), \]

\[ S^*(y_1, \ldots, y_{t-1}) = \arg \min_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)). \]
To play the minimax strategy: after seeing $y_1, \ldots, y_{t-1}$,
Computing minimax optimal strategies

To play the minimax strategy: after seeing $y_1, \ldots, y_{t-1}$,

1. Compute $V$.
Computing minimax optimal strategies

To play the minimax strategy: after seeing $y_1, \ldots, y_{t-1}$,

1. Compute $V$,

2. Choose $a_t$ as the minimizer of

$$\max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t))$$
To play the minimax strategy: after seeing \( y_1, \ldots, y_{t-1} \),

1. Compute \( V \),
2. Choose \( a_t \) as the minimizer of

\[
\max_{y_t} \left( \ell(a_t, y_t) + V(y_1, \ldots, y_t) \right)
\]

Difficult!
Computing minimax optimal strategies

To play the minimax strategy: after seeing $y_1, \ldots, y_{t-1}$,

1. Compute $V$,
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$$\max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t))$$

Difficult!

Efficient minimax optimal strategies

When is $V$ a simple function of (statistics of) the history $y_1, \ldots, y_t$?
Computing minimax optimal strategies.


Part 2: Linear regression.

Time series forecasting.
Online prediction with Euclidean loss

(with Wouter Koolen, Alan Malek)

Prediction in $\mathbb{R}^d$: $Y \subseteq \mathbb{R}^d$, $A = \mathbb{R}^d$, Euclidean loss:

$$\ell(\hat{y}, y) = \frac{1}{2} \| \hat{y} - y \|_2^2.$$ 

Minimax strategy is empirical minimizer plus shrinkage towards center of smallest ball containing $Y$:

$$a^t_{t+1} = t \alpha_{t+1} \bar{y}_t + (1 - t \alpha_{t+1}) c.$$ 

Regret:

$$r^2 T \sum_{t=1}^{T} \alpha_t,$$

where $r$ is radius of smallest ball, $\alpha_T = 1$, $\alpha_t = \alpha_{t+1}^2 + \alpha_{t+1}$.
Online prediction with Euclidean loss
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- Prediction in $\mathbb{R}^d$:
  $\mathcal{Y} \subseteq \mathbb{R}^d$, $\mathcal{A} = \mathbb{R}^d$, Euclidean loss: $\ell(\hat{y}, y) = \frac{1}{2} \|\hat{y} - y\|^2$. 

Minimax strategy is empirical minimizer plus shrinkage towards center of smallest ball containing $\mathcal{Y}$:

$$a^*_t + 1 = t \alpha^*_t + 1 \bar{y}_t + (1 - t \alpha^*_t + 1) c.$$ 

Regret:

$$r^2 = T \sum_{t=1}^T \alpha^*_t,$$

where $r$ is radius of smallest ball, $\alpha^*_T = 1$, $\alpha^*_t = \alpha^*_{t+1}^2 + \alpha^*_t + 1$. 

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Online prediction with Euclidean loss
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- Prediction in $\mathbb{R}^d$:
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- Minimax strategy is empirical minimizer plus shrinkage towards center of smallest ball containing $\mathcal{Y}$: $a_{t+1}^* = t\alpha_{t+1}\bar{y}_t + (1 - t\alpha_{t+1})c$. 

Regret: $r^2 = T\sum_{t=1}^T \alpha_t$, where $r$ is radius of smallest ball, $\alpha_T = 1$, $\alpha_t = \alpha^2_{t+1} + \alpha_{t+1}$. 

\[ r = \frac{1}{2} \|\hat{y} - y\|^2 \]
Online prediction with Euclidean loss
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- Prediction in $\mathbb{R}^d$: $\mathcal{Y} \subseteq \mathbb{R}^d$, $\mathcal{A} = \mathbb{R}^d$, Euclidean loss: $\ell(\hat{y}, y) = \frac{1}{2} \|\hat{y} - y\|^2$.

- Minimax strategy is empirical minimizer plus shrinkage towards center of smallest ball containing $\mathcal{Y}$: $a^*_{t+1} = t\alpha_{t+1}\bar{y}_t + (1 - t\alpha_{t+1})c$.

- Regret:

$$r^2 \sum_{t=1}^{T} \alpha_t,$$

where $r$ is radius of smallest ball,

$$\alpha_T = \frac{1}{T}, \quad \alpha_t = \alpha_{t+1}^2 + \alpha_{t+1}$$
Online prediction with quadratic loss

The simplex case

Suppose $\mathcal{Y}$ is a set of $d + 1$ affinely independent points in $\mathbb{R}^d$, all lying on the surface of the smallest ball.
Online prediction with quadratic loss

The simplex case

Suppose \( \mathcal{Y} \) is a set of \( d + 1 \) affinely independent points in \( \mathbb{R}^d \), all lying on the surface of the smallest ball.

Maintain statistics:
\[
\begin{align*}
    s_n &= \sum_{t=1}^{n} (y_t - c), \\
    \sigma_n^2 &= \sum_{t=1}^{n} \|y_t - c\|^2.
\end{align*}
\]
Online prediction with quadratic loss

The simplex case

Suppose \( \mathcal{Y} \) is a set of \( d + 1 \) affinely independent points in \( \mathbb{R}^d \), all lying on the surface of the smallest ball.

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\[
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\]

Value-to-go: quadratic in state

\[
\frac{1}{2} \left( \alpha_n \|s_n\|^2 - \sigma_n^2 + r^2 \sum_{t=n+1}^{T} \alpha_t \right).
\]
Online prediction with quadratic loss

The simplex case

Suppose $\mathcal{Y}$ is a set of $d + 1$ affinely independent points in $\mathbb{R}^d$, all lying on the surface of the smallest ball.

Maintain statistics:

\[ s_n = \sum_{t=1}^{n} (y_t - c), \quad \sigma_n^2 = \sum_{t=1}^{n} \|y_t - c\|^2. \]

Value-to-go: quadratic in state

\[
\frac{1}{2} \left( \alpha_n \|s_n\|^2 - \sigma_n^2 + r^2 \sum_{t=n+1}^{T} \alpha_t \right).
\]

Maximin distribution: same mean.

\[
\alpha_T = \frac{1}{T}, \quad \alpha_t = \alpha_{t+1}^2 + \alpha_{t+1}
\]
Online prediction with quadratic loss

The simplex case

Suppose $\mathcal{Y}$ is a set of $d + 1$ affinely independent points in $\mathbb{R}^d$, all lying on the surface of the smallest ball. Maintain statistics: $s_n = \sum_{t=1}^{n} (y_t - c)$, $\sigma_n^2 = \sum_{t=1}^{n} \|y_t - c\|^2$.

Value-to-go: quadratic in state

$$\frac{1}{2} \left( \alpha_n \|s_n\|^2 - \sigma_n^2 + r^2 \sum_{t=n+1}^{T} \alpha_t \right).$$

Minimax strategy: affine in state

$$a_{n+1}^* - c = n\alpha_{n+1} \frac{s_n}{n}.$$
Online prediction with quadratic loss

The simplex case

Suppose $\mathcal{Y}$ is a set of $d + 1$ affinely independent points in $\mathbb{R}^d$, all lying on the surface of the smallest ball. Maintain statistics:

$$s_n = \sum_{t=1}^{n} (y_t - c), \quad \sigma_n^2 = \sum_{t=1}^{n} \|y_t - c\|^2.$$  

Value-to-go: quadratic in state

$$\frac{1}{2} \left( \alpha_n \|s_n\|^2 - \sigma_n^2 + r^2 \sum_{t=n+1}^{T} \alpha_t \right).$$

Minimax strategy: affine in state

$$a_{n+1}^* - c = n\alpha_{n+1} \frac{s_n}{n}.$$  

$$a_{n+1}^* = n\alpha_{n+1} \bar{y}_n + (1 - n\alpha_{n+1}) c$$

$$\alpha_T = \frac{1}{T}, \quad \alpha_t = \alpha_{t+1}^2 + \alpha_{t+1}.$$
Online prediction with quadratic loss

The simplex case

Suppose $\mathcal{Y}$ is a set of $d + 1$ affinely independent points in $\mathbb{R}^d$, all lying on the surface of the smallest ball. Maintain statistics: $s_n = \sum_{t=1}^n(y_t - c)$, $\sigma_n^2 = \sum_{t=1}^n \|y_t - c\|^2$.

Value-to-go: quadratic in state

$$\frac{1}{2} \left( \alpha_n \|s_n\|^2 - \sigma_n^2 + r^2 \sum_{t=n+1}^T \alpha_t \right).$$

Minimax strategy: affine in state

$$a_{n+1}^* - c = n\alpha_{n+1} \frac{s_n}{n}.$$  

$$a_{n+1}^* = n\alpha_{n+1} \bar{y}_n + (1 - n\alpha_{n+1})c.$$

$$\alpha_T = \frac{1}{T}, \quad \alpha_t = \alpha_{t+1}^2 + \alpha_{t+1} \leq \frac{1}{t}.$$
Online prediction with quadratic loss

The simplex case

Suppose $Y$ is a set of $d + 1$ affinely independent points in $\mathbb{R}^d$, all lying on the surface of the smallest ball.

Maintain statistics: $s_n = \sum_{t=1}^{n} (y_t - c)$, $\sigma^2_n = \sum_{t=1}^{n} \|y_t - c\|^2$.

Value-to-go: quadratic in state

$$\frac{1}{2} \left( \alpha_n \|s_n\|^2 - \sigma^2_n + r^2 \sum_{t=n+1}^{T} \alpha_t \right).$$

Minimax strategy: affine in state

$$a_{n+1}^* - c = n\alpha_{n+1} \frac{s_n}{n}.$$

$$a_{n+1}^* = n\alpha_{n+1} \bar{y}_n + (1 - n\alpha_{n+1})c$$

Maximin distribution: same mean.

$$\alpha_T = \frac{1}{T}, \quad \alpha_t = \alpha_{t+1}^2 + \alpha_{t+1} \leq \frac{1}{t}.$$
Proof idea

\[ V(y_1, \ldots, y_T) := - \min_{a} \sum_{t=1}^{T} \ell(a, y_t), \]

\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} \left( \ell(a_t, y_t) + V(y_1, \ldots, y_{t-1}) \right). \]

The final value function is a (convex) quadratic in the state.

\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} E_{p_t \sim y_t} \left( \ell(a_t, y_t) + V(y_1, \ldots, y_{t-1}) \right) = \max_{p_t} \min_{a_t} E_{y_t \sim p_t} \left( \ell(a_t, y_t) + V(y_1, \ldots, y_{t-1}) \right). \]

At each step, the unconstrained maximizer in \( \{ p \in \mathbb{R}^{d+1} : 1^\top p = 1 \} \) keeps the value-to-go a quadratic function. When the simplex points are on the surface of the smallest ball, the maximizer is a probability distribution.
Proof idea

\[ V(y_1, \ldots, y_T) := -\min_a \sum_{t=1}^{T} \ell(a, y_t), \]
\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)). \]
Online prediction with quadratic loss on the simplex

Proof idea

\[ V(y_1, \ldots, y_T) := -\min_a \sum_{t=1}^{T} \ell(a, y_t), \]

\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} \left( \ell(a_t, y_t) + V(y_1, \ldots, y_t) \right). \]

The final \( V(y_1, \ldots, y_T) \) is a (convex) quadratic in the state.
Online prediction with quadratic loss on the simplex

Proof idea

\[ V(y_1, \ldots, y_T) := - \min_a \sum_{t=1}^T \ell(a, y_t), \]

\[ V(y_1, \ldots, y_{t-1}) := \min \max_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)). \]

The final \( V(y_1, \ldots, y_T) \) is a (convex) quadratic in the state.

\[ V(y_1, \ldots, y_{t-1}) := \min \max_{a_t} \mathbb{E}_{y_t \sim p_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)) \]
Online prediction with quadratic loss on the simplex

Proof idea

\[ V(y_1, \ldots, y_T) := -\min_a \sum_{t=1}^{T} \ell(a, y_t), \]

\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)). \]

The final \( V(y_1, \ldots, y_T) \) is a (convex) quadratic in the state.

\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{p_t} \mathbb{E}_{y_t \sim p_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)) \]

\[ = \max_{p_t} \min_{a_t} \mathbb{E}_{y_t \sim p_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)). \]
Online prediction with quadratic loss on the simplex

Proof idea

\[ V(y_1, \ldots, y_T) := -\min_a \sum_{t=1}^{T} \ell(a, y_t), \]

\[ V(y_1, \ldots, y_{t-1}) := \min \max_{a_t, y_t} \left( \ell(a_t, y_t) + V(y_1, \ldots, y_t) \right). \]

The final \( V(y_1, \ldots, y_T) \) is a (convex) quadratic in the state.

\[ V(y_1, \ldots, y_{t-1}) := \min \max_{a_t, p_t} \mathbb{E}_{y_t \sim p_t} \left( \ell(a_t, y_t) + V(y_1, \ldots, y_t) \right) \]

\[ = \max \min_{p_t, a_t} \mathbb{E}_{y_t \sim p_t} \left( \ell(a_t, y_t) + V(y_1, \ldots, y_t) \right). \]

At each step, the unconstrained maximizer in \( \{p \in \mathbb{R}^{d+1} : 1^\top p = 1\} \) keeps the value-to-go a quadratic function.
Online prediction with quadratic loss on the simplex

Proof idea

\[ V(y_1, \ldots, y_T) := -\min_a \sum_{t=1}^T \ell(a, y_t), \]

\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_T)). \]

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\[ = \max_{p_t} \min_{a_t} \mathbb{E}_{y_t \sim p_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_T)). \]

At each step, the unconstrained maximizer in \( \{p \in \mathbb{R}^{d+1} : 1^\top p = 1\} \) keeps the value-to-go a quadratic function.

When the simplex points are on the surface of the smallest ball, the maximizer is a probability distribution.
Online prediction with quadratic loss on the ball

The ball case: $\mathcal{Y} = \{y : \|y - c\| \leq r\}$

- Maintain statistics: $s_n = \sum_{t=1}^n (y_t - c)$, $\sigma_n^2 = \sum_{t=1}^n \|y_t - c\|^2$.
- Value-to-go: quadratic in state $\frac{1}{2} (\alpha_n \|s_n\|^2 - \sigma_n^2 + r^2 T \sum_{t=n+1}^\infty \alpha_t)$.
- Minimax strategy: affine in state $a^*_{n+1} - c = \alpha_{n+1} s_n$.
- Maximin distribution: same mean.
- Minimax regret for ball $V(\mathcal{Y}) = \frac{r^2}{2 T \sum_{t=1}^\infty \alpha_t}$. 
Online prediction with quadratic loss on the ball

The ball case: \( \mathcal{Y} = \{y : \|y - c\| \leq r\} \)

Maintain statistics: 
\[
\begin{align*}
    s_n &= \sum_{t=1}^{n} (y_t - c), \\
    \sigma_n^2 &= \sum_{t=1}^{n} \|y_t - c\|^2.
\end{align*}
\]
Online prediction with quadratic loss on the ball

The ball case: $\mathcal{Y} = \{ y : \|y - c\| \leq r \}$

Maintain statistics: $s_n = \sum_{t=1}^{n}(y_t - c)$, $\sigma_n^2 = \sum_{t=1}^{n}\|y_t - c\|^2$.

Value-to-go: quadratic in state

$$\frac{1}{2} \left( \alpha_n \|s_n\|^2 - \sigma_n^2 + r^2 \sum_{t=n+1}^{T} \alpha_t \right).$$
Online prediction with quadratic loss on the ball

The ball case: \( \mathcal{Y} = \{ y : \| y - c \| \leq r \} \)

Maintain statistics: \( s_n = \sum_{t=1}^{n} (y_t - c) \), \( \sigma_n^2 = \sum_{t=1}^{n} \| y_t - c \|^2 \).

Value-to-go: quadratic in state

\[
\frac{1}{2} \left( \alpha_n \| s_n \|^2 - \sigma_n^2 + r^2 \sum_{t=n+1}^{T} \alpha_t \right).
\]

Minimax strategy: affine in state

\[
a_{n+1}^* - c = n \alpha_{n+1} \frac{s_n}{n}.
\]
Online prediction with quadratic loss on the ball

The ball case: \( \mathcal{Y} = \{ y : \| y - c \| \leq r \} \)

Maintain statistics: \( s_n = \sum_{t=1}^{n} (y_t - c), \quad \sigma_n^2 = \sum_{t=1}^{n} \| y_t - c \|^2. \)

Value-to-go: quadratic in state

\[
\frac{1}{2} \left( \alpha_n \| s_n \|^2 - \sigma_n^2 + r^2 \sum_{t=n+1}^{T} \alpha_t \right).
\]

Minimax strategy: affine in state

\[
a_{n+1}^* - c = n\alpha_{n+1} \frac{s_n}{n},
\]

\[
a_{n+1}^* = n\alpha_{n+1} \bar{y}_n + (1 - n\alpha_{n+1}) c
\]
Online prediction with quadratic loss on the ball

### The ball case: \( \mathcal{Y} = \{ y : \| y - c \| \leq r \} \)

Maintain statistics:

\[
\begin{align*}
    s_n &= \sum_{t=1}^{n} (y_t - c), \\
    \sigma_n^2 &= \sum_{t=1}^{n} \| y_t - c \|^2.
\end{align*}
\]

### Value-to-go: quadratic in state

\[
\frac{1}{2} \left( \alpha_n \| s_n \|^2 - \sigma_n^2 + r^2 \sum_{t=n+1}^{T} \alpha_t \right).
\]

### Minimax strategy: affine in state

\[
\begin{align*}
a^*_{n+1} - c &= n\alpha_{n+1} \frac{s_n}{n}, \\
a^*_{n+1} &= n\alpha_{n+1} \bar{y}_n + (1 - n\alpha_{n+1}) c.
\end{align*}
\]

Maximin distribution: same mean.
Online prediction with quadratic loss on the ball

The ball case: \( \mathcal{Y} = \{ y : \| y - c \| \leq r \} \)

Maintain statistics: 
\[
s_n = \sum_{t=1}^{n} (y_t - c), \quad \sigma_n^2 = \sum_{t=1}^{n} \| y_t - c \|^2.
\]

Value-to-go: quadratic in state
\[
\frac{1}{2} \left( \alpha_n \| s_n \|^2 - \sigma_n^2 + r^2 \sum_{t=n+1}^{T} \alpha_t \right).
\]

Minimax strategy: affine in state
\[
a_{n+1}^* - c = n \alpha_{n+1} \frac{s_n}{n}.
\]
\[
a_{n+1}^* = n \alpha_{n+1} \bar{y}_n + (1 - n \alpha_{n+1}) c
\]

Maximin distribution: same mean.

Minimax regret for ball
\[
V(\mathcal{Y}) = \frac{r^2}{2} \sum_{t=1}^{T} \alpha_t.
\]
Proof idea

Online prediction with quadratic loss on the ball
Proof idea

\[
V(y_1, \ldots, y_T) := -\min_a \sum_{t=1}^T \ell(a, y_t),
\]

\[
V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)).
\]
Online prediction with quadratic loss on the ball

Proof idea

\[ V(y_1, \ldots, y_T) := -\min_a \sum_{t=1}^T \ell(a, y_t), \]

\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)). \]

The final \( V(y_1, \ldots, y_T) \) is a (convex) quadratic in the state.
Proof idea

\[ V(y_1, \ldots, y_T) := -\min_a \sum_{t=1}^{T} \ell(a, y_t), \]

\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)). \]

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\[ V(y_1, \ldots, y_T) := -\min_a \sum_{t=1}^T \ell(a, y_t), \]

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The final \( V(y_1, \ldots, y_T) \) is a (convex) quadratic in the state.

\[ V(y_1, \ldots, y_{t-1}) := \min_{a_t} \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)) \]

At each step, the inner maximum is of a (convex) quadratic criterion with a single quadratic constraint. This is a rare example of a nonconvex problem where strong duality holds.
Online prediction with quadratic loss on the ball

Proof idea

\[ V(y_1, \ldots, y_T) := -\min_a \sum_{t=1}^T \ell(a, y_t), \]

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\[ V(y_1, \ldots, y_{t-1}) := \min_a \max_{y_t} (\ell(a_t, y_t) + V(y_1, \ldots, y_t)). \]

At each step, the inner maximum is of a (convex) quadratic criterion with a single quadratic constraint. This is a rare example of a nonconvex problem where strong duality holds. Evaluating the dual gives the recurrence for the value-to-go.
The general case: closed, bounded $\mathcal{Y} \subset \mathbb{R}^d$

Recall: the smallest ball containing $\mathcal{Y}$ is $B_{\mathcal{Y}} = \{ x \in \mathbb{R}^d : \| x - c \| \leq r \}$.

A Lagrange dual argument shows that the optimal center is in the convex hull of a set of contact points of $\mathcal{Y}$ at radius $r$.

From Carathéodory's Theorem, there is an affinely independent subset $S$ of these contact points, with $|S| \leq d + 1$.

From below $\mathcal{Y} \supseteq S$, so $V(\mathcal{Y}) \geq V(S) = r^2 \sum_{i=1}^{\alpha i} \alpha_i$.

From above $\mathcal{Y} \subseteq B_{\mathcal{Y}}$, so $V(\mathcal{Y}) \leq V(B_{\mathcal{Y}}) = r^2 \sum_{i=1}^{\alpha i} \alpha_i$. 


Online prediction with quadratic loss

The general case: closed, bounded $\mathcal{Y} \subset \mathbb{R}^d$

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From below $\mathcal{Y} \supseteq S$, so $V(\mathcal{Y}) \geq V(S) = \frac{1}{2} \sum_{i=1}^{r} \alpha_i$.

From above $\mathcal{Y} \subseteq B_\mathcal{Y}$, so $V(\mathcal{Y}) \leq V(B_\mathcal{Y}) = \frac{1}{2} \sum_{i=1}^{r} \alpha_i$. 
The general case: closed, bounded $\mathcal{Y} \subset \mathbb{R}^d$

Recall: the smallest ball containing $\mathcal{Y}$ is $B_{\mathcal{Y}} = \{ x \in \mathbb{R}^d : \| x - c \| \leq r \}$. A Lagrange dual argument shows that the optimal center is in the convex hull of a set of contact points of $\mathcal{Y}$ at radius $r$. 
Online prediction with quadratic loss

The general case: closed, bounded $\mathcal{Y} \subset \mathbb{R}^d$

Recall: the smallest ball containing $\mathcal{Y}$ is $B_{\mathcal{Y}} = \{x \in \mathbb{R}^d : \|x - c\| \leq r\}$. A Lagrange dual argument shows that the optimal center is in the convex hull of a set of contact points of $\mathcal{Y}$ at radius $r$. From Carathéodory’s Theorem, there is an affinely independent subset $S$ of these contact points, with $|S| \leq d + 1$. 
Online prediction with quadratic loss

The general case: closed, bounded $\mathcal{Y} \subset \mathbb{R}^d$

Recall: the smallest ball containing $\mathcal{Y}$ is $B_\mathcal{Y} = \{ x \in \mathbb{R}^d : \| x - c \| \leq r \}$.
A Lagrange dual argument shows that the optimal center is in the convex hull of a set of contact points of $\mathcal{Y}$ at radius $r$.
From Carathéodory’s Theorem, there is an affinely independent subset $S$ of these contact points, with $|S| \leq d + 1$.

From below

$\mathcal{Y} \supseteq S$, so

$$V(\mathcal{Y}) \geq V(S) = \frac{r^2}{2} \sum_{i=1}^{T} \alpha_i.$$
Online prediction with quadratic loss

**The general case: closed, bounded** $\mathcal{Y} \subset \mathbb{R}^d$

Recall: the smallest ball containing $\mathcal{Y}$ is $B_\mathcal{Y} = \{x \in \mathbb{R}^d : \|x - c\| \leq r\}$. A Lagrange dual argument shows that the optimal center is in the convex hull of a set of contact points of $\mathcal{Y}$ at radius $r$. From Carathéodory’s Theorem, there is an affinely independent subset $S$ of these contact points, with $|S| \leq d + 1$.

**From below**

$\mathcal{Y} \supseteq S$, so

$$V(\mathcal{Y}) \geq V(S) = \frac{r^2}{2} \sum_{i=1}^{T} \alpha_i.$$ 

**From above**

$\mathcal{Y} \subseteq B_\mathcal{Y}$, so

$$V(\mathcal{Y}) \leq V(B_\mathcal{Y}) = \frac{r^2}{2} \sum_{i=1}^{T} \alpha_i.$$
Main result: the role of the smallest ball

The smallest ball: $B_Y$

The smallest ball containing $Y$ is $B_Y = \{ y \in \mathbb{R}^d : \| y - c \| \leq r \}$, with $c = \arg \min_c \max_{y \in Y} \| y - c \|$, $r = \min_c \max_{y \in Y} \| y - c \|$. 

Main Theorem

For closed, bounded $Y \subset \mathbb{R}^d$:

Minimax strategy is $a_{n+1}^* = n\alpha_{n+1} \frac{1}{n} \sum_{t=1}^{n} y_t + (1 - n\alpha_{n+1})c$.

Optimal regret is $V(Y) = \frac{r^2}{2} \sum_{n=1}^{T} \alpha_n$. 
Online prediction with quadratic loss

Minimax regret

\[ V(\mathcal{Y}) = \frac{r^2}{2} \sum_{t=1}^{T} \alpha_t \]
Online prediction with quadratic loss

Minimax regret

\[ V(Y) = \frac{r^2}{2} \sum_{t=1}^{T} \alpha_t = \frac{r^2}{2} \left( \log T - \log \log T + O \left( \frac{\log \log T}{\log T} \right) \right). \]
Online prediction with quadratic loss

\[ V(\mathcal{Y}) = \frac{r^2}{2} \sum_{t=1}^{T} \alpha_t = \frac{r^2}{2} \left( \log T - \log \log T + O\left(\frac{\log \log T}{\log T}\right) \right). \]
Computing minimax optimal strategies.


Part 2: Linear regression.
  - Fixed design.
  - Minimax strategy is regularized least squares.
  - Box and ellipsoid constraints.
  - Adversarial covariates.

Time series forecasting.
Online fixed design linear regression
(with Koolen, Malek, Takimoto, Warmuth)

Given:
\[ T; \mathbf{x}_1, \ldots, \mathbf{x}_T \in \mathbb{R}^p; \mathbf{y}_1, \ldots, \mathbf{y}_T \subseteq \mathbb{R}^T. \]

For \( t = 1, 2, \ldots, T \):
- Learner predicts \( \hat{y}_t \in \mathbb{R}^p \)
- Adversary reveals \( y_t \in \mathbb{R}^p \) \((y_1, \ldots, y_T) \subseteq \mathbb{R}^T\)
- Learner incurs loss \( (\hat{y}_t - y_t)^2 \)

Regret = \[ T \sum_{t=1}^{T} (\hat{y}_t - y_t)^2 - \min_{\beta \in \mathbb{R}^p} T \sum_{t=1}^{T} (\beta^\top \mathbf{x}_t - y_t)^2. \]
Online fixed design linear regression
(with Koolen, Malek, Takimoto, Warmuth)

Protocol

Given: $T; x_1, \ldots, x_T \in \mathbb{R}^p$; $Y_T \subset \mathbb{R}^T$.

For $t = 1, 2, \ldots, T$:
- Learner predicts $\hat{y}_t \in \mathbb{R}$
- Adversary reveals $y_t \in \mathbb{R}^T (y_T^t \in Y_T)$
- Learner incurs loss $(\hat{y}_t - y_t)^2$.

Regret $= T \sum_{t=1}^T (\hat{y}_t - y_t)^2 - \min_{\beta \in \mathbb{R}^p} T \sum_{t=1}^T (\beta^\top x_t - y_t)^2$. 
Online fixed design linear regression
(with Koolen, Malek, Takimoto, Warmuth)

Protocol

Given: $T$;

Learner predicts $\hat{y}_t \in \mathbb{R}^p$
Adversary reveals $y_t \in \mathbb{R}^p (y_T \in Y_T)$
Learner incurs loss $(\hat{y}_t - y_t)^2$

Regret = $\sum_{t=1}^{T} (\hat{y}_t - y_t)^2 - \min_{\beta \in \mathbb{R}^p} \sum_{t=1}^{T} (\beta^\top x_t - y_t)^2$
Protocol

Given: \( T; x_1, \ldots, x_T \in \mathbb{R}^p; \)
Protocol

Given: $T; x_1, \ldots, x_T \in \mathbb{R}^p; Y^T \subset \mathbb{R}^T$. 

\[
\text{Regret} = \sum_{t=1}^{T} (\hat{y}_t - y_t)^2 - \min_{\beta \in \mathbb{R}^p} \sum_{t=1}^{T} (\beta^\top x_t - y_t)^2.
\]
Online fixed design linear regression
(with Koolen, Malek, Takimoto, Warmuth)

Protocol

Given: \( T ; x_1, \ldots, x_T \in \mathbb{R}^p ; \mathcal{Y}^T \subset \mathbb{R}^T. \)
For \( t = 1, 2, \ldots, T \) :

1. Learner predicts \( \hat{y}_t \in \mathbb{R} \).
2. Adversary reveals \( y_t \in \mathcal{Y}^T \).
3. Learner incurs loss \((\hat{y}_t - y_t)^2\).

Regret = \( T \sum_{t=1}^T (\hat{y}_t - y_t)^2 - \min_{\beta \in \mathbb{R}^p} T \sum_{t=1}^T (\beta^\top x_t - y_t)^2. \)
Online fixed design linear regression  
(with Kolen, Malek, Takimoto, Warmuth)

Protocol

Given: \( T; x_1, \ldots, x_T \in \mathbb{R}^p; Y_T \subset \mathbb{R}^T. \)

For \( t = 1, 2, \ldots, T \):

- Learner predicts \( \hat{y}_t \in \mathbb{R} \)
Online fixed design linear regression
(with Koolen, Malek, Takimoto, Warmuth)

Protocol

Given: $T; x_1, \ldots, x_T \in \mathbb{R}^p; \mathcal{Y}^T \subset \mathbb{R}^T$.
For $t = 1, 2, \ldots, T$:

- Learner predicts $\hat{y}_t \in \mathbb{R}$
- Adversary reveals $y_t \in \mathbb{R}$

Regret $= \sum_{t=1}^{T} (\hat{y}_t - y_t)^2 - \min_{\beta \in \mathbb{R}^p} \sum_{t=1}^{T} (\beta^\top x_t - y_t)^2$. 
Protocol

Given: $T; x_1, \ldots, x_T \in \mathbb{R}^p; \mathcal{Y}^T \subset \mathbb{R}^T$.

For $t = 1, 2, \ldots, T$ :

- Learner predicts $\hat{y}_t \in \mathbb{R}$
- Adversary reveals $y_t \in \mathbb{R}$ ($y_1^T \in \mathcal{Y}^T$)
Online fixed design linear regression
(with Koolen, Malek, Takimoto, Warmuth)

Protocol

Given: $T; x_1, \ldots, x_T \in \mathbb{R}^p; \mathcal{Y}^T \subset \mathbb{R}^T$.
For $t = 1, 2, \ldots, T$:
- Learner predicts $\hat{y}_t \in \mathbb{R}$
- Adversary reveals $y_t \in \mathbb{R}$ ($y_1^T \in \mathcal{Y}^T$)
- Learner incurs loss $(\hat{y}_t - y_t)^2$. 

Regret $= \sum_{t=1}^{T} (\hat{y}_t - y_t)^2 - \min_{\beta} \sum_{t=1}^{T} (\beta^\top x_t - y_t)^2$. 

Online fixed design linear regression
(with Koolen, Malek, Takimoto, Warmuth)

Protocol

Given: \( T; x_1, \ldots, x_T \in \mathbb{R}^p; \mathcal{Y}^T \subset \mathbb{R}^T \).

For \( t = 1, 2, \ldots, T \):

- Learner predicts \( \hat{y}_t \in \mathbb{R} \)
- Adversary reveals \( y_t \in \mathbb{R} \) (\( y_1^T \in \mathcal{Y}^T \))
- Learner incurs loss \( (\hat{y}_t - y_t)^2 \).

Regret = \( \sum_{t=1}^{T} (\hat{y}_t - y_t)^2 - \min_{\beta \in \mathbb{R}^p} \sum_{t=1}^{T} \left( \beta^\top x_t - y_t \right)^2 \).
Online fixed design linear regression

Online linear regression: previous work

- (Foster, 1991): $\ell_2$-regularized least squares.
- (Cesa-Bianchi et al, 1996): $\ell_2$-constrained least squares.
- (Kivinen and Warmuth, 1997): exponentiated gradient (relative entropy regularization).
- (Forster, 1999; Azoury and Warmuth, 2001): aggregating algorithm is last-step minimax.
Linear regression in a probabilistic setting

Ordinary least squares (linear model, uncorrelated errors)

Given \((x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^p \times \mathbb{R}\),

\[
\hat{\beta} = \left( \sum_{t=1}^{n} x_t x_t^\top \right)^{-1} \left( \sum_{t=1}^{n} x_t y_t \right),
\]

and for a subsequent \(x \in \mathbb{R}^p\), predict

\[
\hat{y} = x^\top \hat{\beta} = x^\top \left( \sum_{t=1}^{n} x_t x_t^\top \right)^{-1} \left( \sum_{t=1}^{n} x_t y_t \right).
\]
Linear regression in a probabilistic setting

Ordinary least squares (linear model, uncorrelated errors)

Given \((x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^p \times \mathbb{R}\), choose

\[
\hat{\beta} = \left( \sum_{t=1}^{n} x_t x_t^T \right)^{-1} \sum_{t=1}^{n} x_t y_t,
\]

A sequential version of OLS

\[
\hat{y}_{n+1} = x_{n+1}^T \left( \sum_{t=1}^{n} x_t x_t^T \right)^{-1} \sum_{t=1}^{n} x_t y_t.
\]
## Ordinary least squares (linear model, uncorrelated errors)

Given \((x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^p \times \mathbb{R}\), choose

\[
\hat{\beta} = \left( \sum_{t=1}^{n} x_t x_t^\top \right)^{-1} \sum_{t=1}^{n} x_t y_t,
\]

and for a subsequent \(x \in \mathbb{R}^p\), predict

\[
\hat{y} = x^\top \hat{\beta} = x^\top \left( \sum_{t=1}^{n} x_t x_t^\top \right)^{-1} \sum_{t=1}^{n} x_t y_t,
\]
Linear regression in a probabilistic setting

### Ordinary least squares

Given \((x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^p \times \mathbb{R}\), choose

\[
\hat{\beta} = \left( \sum_{t=1}^{n} x_t x_t^\top \right)^{-1} \sum_{t=1}^{n} x_t y_t,
\]

and for a subsequent \(x \in \mathbb{R}^p\), predict

\[
\hat{y} = x^\top \hat{\beta} = x^\top \left( \sum_{t=1}^{n} x_t x_t^\top \right)^{-1} \sum_{t=1}^{n} x_t y_t,
\]

### A sequential version of OLS

\[
\hat{y}_{n+1} := x_{n+1}^\top \left( \sum_{t=1}^{n} x_t x_t^\top \right)^{-1} \sum_{t=1}^{n} x_t y_t.
\]
### Ordinary least squares (linear model, uncorrelated errors)

Given \((x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^p \times \mathbb{R}\), choose

\[
\hat{\beta} = \left( \sum_{t=1}^{n} x_t x_t^\top \right)^{-1} \sum_{t=1}^{n} x_t y_t,
\]

and for a subsequent \(x \in \mathbb{R}^p\), predict

\[
\hat{y} = x^\top \hat{\beta} = x^\top \left( \sum_{t=1}^{n} x_t x_t^\top \right)^{-1} \sum_{t=1}^{n} x_t y_t,
\]

### A sequential version of ridge regression

\[
\hat{y}_{n+1} := x_{n+1}^\top \left( \sum_{t=1}^{n} x_t x_t^\top + \lambda I \right)^{-1} \sum_{t=1}^{n} x_t y_t.
\]
Fix $x_1, \ldots, x_T \in \mathbb{R}^p$. 

Sufficient statistics

Value-to-go: quadratic

Minimax strategy: linear

Maximin distribution:
Fix $x_1, \ldots, x_T \in \mathbb{R}^p$. 

$\mathcal{Y}^T = \{(y_1, \ldots, y_T) : |y_t| \leq B_t\}$. 
Online fixed design linear regression

### Sufficient statistics

Fix $x_1, \ldots, x_T \in \mathbb{R}^p$.

Maintain statistics: $s_n = \sum_{t=1}^{n} y_t x_t$

$\mathcal{Y}^T = \{(y_1, \ldots, y_T) : |y_t| \leq B_t\}$.
Online fixed design linear regression

Sufficient statistics

Fix $x_1, \ldots, x_T \in \mathbb{R}^p$. Maintain statistics: $s_n = \sum_{t=1}^{n} y_t x_t$

$\mathcal{Y}^T = \{(y_1, \ldots, y_T) : |y_t| \leq B_t\}$.

Minimax* strategy: linear

$\hat{y}_{n+1}^* = x_{n+1}^TP_{n+1}s_n$. 

Online fixed design linear regression

### Sufficient statistics

Fix $x_1, \ldots, x_T \in \mathbb{R}^p$.

- Maintain statistics: $s_n = \sum_{t=1}^n y_t x_t$

- $\mathcal{Y}^T = \{(y_1, \ldots, y_T) : |y_t| \leq B_t\}$.

### Minimax* strategy: linear

$$\hat{y}_{n+1}^* = x_{n+1} P_{n+1} s_n.$$
Online fixed design linear regression

**Sufficient statistics**

Fix \( x_1, \ldots, x_T \in \mathbb{R}^p \).

Maintain statistics: \( s_n = \sum_{t=1}^{n} y_t x_t \)

\( \mathcal{Y}^T = \{(y_1, \ldots, y_T) : |y_t| \leq B_t\} \).

**Minimax* strategy: linear**

\[
\hat{y}_{n+1}^* = x_{n+1}^T P_{n+1} s_n.
\]

\[
P_{n+1}^{-1} = \sum_{t=1}^{n} x_t x_t^T + \sum_{t=n+1}^{T} \frac{x_t^T P_t x_t}{1 + x_t^T P_t x_t} x_t x_t^T.
\]
Online fixed design linear regression

**Sufficient statistics**

Fix \( x_1, \ldots, x_T \in \mathbb{R}^p \).

Maintain statistics: 
\[
\begin{aligned}
  s_n &= \sum_{t=1}^{n} y_t x_t \\
  \sigma^2_n &= \sum_{t=1}^{n} y_t^2
\end{aligned}
\]

\[
\mathcal{Y}^T = \{(y_1, \ldots, y_T) : |y_t| \leq B_t\}.
\]

**Minimax* strategy: linear**

\[
\hat{y}_{n+1}^* = x_{n+1} P_{n+1} s_n.
\]

Maximin distribution:

\[
\Pr (\pm B_{n+1}) = \frac{1}{2} \pm \frac{x_{n+1} P_{n+1} s_n}{2B_{n+1}}.
\]

\[
P_n^{-1} = \sum_{t=1}^{n} x_t x_t^\top + \sum_{t=n+1}^{T} \frac{x_t P_t x_t}{1 + x_t^\top P_t x_t} x_t x_t^\top.
\]
Online fixed design linear regression

**Sufficient statistics**

Fix $x_1, \ldots, x_T \in \mathbb{R}^p$.

Maintain statistics: $s_n = \sum_{t=1}^{n} y_t x_t$

$\mathcal{Y}^T = \{(y_1, \ldots, y_T) : |y_t| \leq B_t\}$.

**Value-to-go: quadratic**

$$s_n \top P_n s_n - \sigma_n^2 + \sum_{t=n+1}^{T} B_t^2 x_t \top P_t x_t.$$

**Minimax* strategy: linear**

$$\hat{y}_{n+1} = x_{n+1} P_{n+1} s_n.$$

Maximin distribution:

$$\Pr (\pm B_{n+1}) = \frac{1}{2} \pm \frac{x_{n+1} P_{n+1} s_n}{2 B_{n+1}}.$$

$$P_n^{-1} = \sum_{t=1}^{n} x_t x_t \top + \sum_{t=n+1}^{T} \frac{x_t \top P_t x_t}{1 + x_t \top P_t x_t} x_t x_t \top.$$
Online fixed design linear regression

Sufficient statistics

Fix $x_1, \ldots, x_T \in \mathbb{R}^p$.

Maintain statistics: $s_n = \sum_{t=1}^{n} y_t x_t$, $\gamma^T = \{(y_1, \ldots, y_T) : |y_t| \leq B_t\}$. $\sigma_n^2 = \sum_{t=1}^{n} y_t^2$.

Value-to-go: quadratic

$$s_n^\top P_n s_n - \sigma_n^2 + \sum_{t=n+1}^{T} B_t^2 x_t^\top P_t x_t.$$ 

Minimax* strategy: linear

$$\hat{y}_{n+1}^* = x_{n+1}^\top P_{n+1} s_n.$$ 

Maximin distribution:

$$\Pr (\pm B_{n+1}) = \frac{1}{2} \pm \frac{x_{n+1}^\top P_{n+1} s_n}{2B_{n+1}}.$$ 

$$P_n^{-1} = \sum_{t=1}^{n} x_t x_t^\top + \sum_{t=n+1}^{T} \frac{x_t^\top P_t x_t}{1 + x_t^\top P_t x_t} x_t x_t^\top.$$
Online fixed design linear regression

**Sufficient statistics**

Fix $x_1, \ldots, x_T \in \mathbb{R}^p$.

Maintain statistics: $s_n = \sum_{t=1}^{n} y_t x_t$, $\gamma^T = \{(y_1, \ldots, y_T) : |y_t| \leq B_t\}$, $\sigma_n^2 = \sum_{t=1}^{n} y_t^2$.

**Value-to-go: quadratic**

$$s_n^\top P_n s_n - \sigma_n^2 + \sum_{t=n+1}^{T} B_t^2 x_t^\top P_t x_t.$$  

* provided: $B_n \geq \sum_{t=1}^{n-1} |x_n^\top P_n x_t| B_t$.

**Minimax* strategy: linear**

$$\hat{y}_{n+1}^* = x_{n+1}^\top P_{n+1} s_n.$$  

Maximin distribution:

$$\Pr (\pm B_{n+1}) = \frac{1}{2} \pm \frac{x_{n+1}^\top P_{n+1} s_n}{2B_{n+1}}.$$  

$$P_n^{-1} = \sum_{t=1}^{n} x_t x_t^\top + \sum_{t=n+1}^{T} \frac{x_t^\top P_t x_t}{1 + x_t^\top P_t x_t} x_t x_t^\top.$$
Linear regression

Box constraints

\[ \mathcal{Y}^T = \{(y_1, \ldots, y_T) : |y_n| \leq B_n\} \]

\[ B_n \geq \sum_{t=1}^{n-1} \left| x_n^\top P_n x_t \right| B_t. \]
**Linear regression**

**Box constraints**

\[ Y^T = \{(y_1, \ldots, y_T) : |y_n| \leq B_n\} \]

\[ B_n \geq \sum_{t=1}^{n-1} \left| x_n^T P_n x_t \right| B_t. \]

**Minimax strategy: linear**

\[ \hat{y}_n^* = x_n^T P_n s_{n-1}. \]
Box constraints

\[ \mathcal{Y}^T = \{(y_1, \ldots, y_T) : |y_n| \leq B_n\} \quad \text{for} \quad B_n \geq \sum_{t=1}^{n-1} \left| x_n^\top P_n x_t \right| B_t. \]

Minimax strategy: linear

\[ \hat{y}^*_n = x_n^\top P_n s_{n-1}. \]

\[ P_n^{-1} = \sum_{t=1}^{n} x_t x_t^\top + \sum_{t=n+1}^{T} \frac{x_t^\top P_t x_t}{1 + x_t^\top P_t x_t} x_t x_t^\top. \]
Linear regression

**Box constraints**

\[ Y^T = \{ (y_1, \ldots, y_T) : |y_n| \leq B_n \} \]

\[ B_n \geq \sum_{t=1}^{n-1} \left| x_n^\top P_n x_t \right| B_t. \]

**Regret**

\[ \text{Regret} = \sum_{t=1}^T B_t^2 x_t^\top P_t x_t. \]

**Minimax strategy: linear**

\[ \hat{y}_n^* = x_n^\top P_n s_{n-1}. \]

**Optimal shrinkage**

\[ P_n^{-1} = \sum_{t=1}^n x_t x_t^\top + \sum_{t=n+1}^T \frac{x_t^\top P_t x_t}{1 + x_t^\top P_t x_t} x_t x_t^\top. \]

c.f. ridge regression:

\[ n \sum_{t=1}^n x_t x_t^\top + \lambda I. \]
Linear regression

Box constraints

\[ Y^T = \{(y_1, \ldots, y_T) : |y_n| \leq B_n \} \]
\[ B_n \geq \sum_{t=1}^{n-1} |x_n^\top P_n x_t| B_t. \]

Regret

\[ \text{Regret} = \sum_{t=1}^{T} B_t^2 x_t^\top P_t x_t. \]

Minimax strategy: linear

\[ \hat{y}_n^* = x_n^\top P_n s_{n-1}. \]

Optimal shrinkage

\[ P_n^{-1} = \sum_{t=1}^{n} x_t x_t^\top + \sum_{t=n+1}^{T} \frac{x_t^\top P_t x_t}{1 + x_t^\top P_t x_t} x_t x_t^\top. \]

\[ \sum_{t=1}^{n} x_t x_t^\top + \lambda I. \]

c.f. ridge regression:
Linear regression

Box constraints

\[ \mathcal{Y}^T = \{ (y_1, \ldots, y_T) : |y_n| \leq B_n \} \quad \text{and} \quad B_n \geq \sum_{t=1}^{n-1} \left| x_n^\top P_n x_t \right| B_t. \]

Minimax strategy: linear

\[ \hat{y}_n^* = x_n^\top P_n s_{n-1}. \]

Optimal shrinkage

\[ P_n^{-1} = \sum_{t=1}^{n} x_t x_t^\top + \sum_{t=n+1}^{T} \frac{x_t^\top P_t x_t}{1 + x_t^\top P_t x_t} x_t x_t^\top. \]

c.f. ridge regression:

\[ \sum_{t=1}^{n} x_t x_t^\top + \lambda I. \]
Linear regression

Box constraints

\[ \mathcal{Y}^T = \{(y_1, \ldots, y_T) : |y_n| \leq B_n\} \]

\[ B_n \geq \sum_{t=1}^{n-1} \left| x_n^\top P_n x_t \right| B_t. \]

Minimax strategy: linear

\[ \hat{y}_n^* = x_n^\top P_n s_{n-1}. \]

Regret

\[ \text{Regret} = \sum_{t=1}^{T} B_t^2 x_t^\top P_t x_t. \]

Optimal shrinkage

\[ P_n^{-1} = \sum_{t=1}^{n} x_t x_t^\top + \sum_{t=n+1}^{T} \frac{x_t^\top P_t x_t}{1 + x_t^\top P_t x_t} x_t x_t^\top. \]

\[ \sum_{t=1}^{n} x_t x_t^\top + \lambda I. \]

c.f. ridge regression:

\[ \sum_{t=1}^{n} x_t x_t^\top + \lambda I. \]
Linear regression: Regret

**Theorem**

\[
\max_{x_1, \ldots, x_T} \sum_{t=1}^{T} x_t^\top P_t x_t \leq p \left( 1 + 2 \ln \left( 1 + \frac{T}{2} \right) \right).
\]
Linear regression

Ellipsoid constraints

$$\mathcal{Y}_R^T = \left\{ (y_1, \ldots, y_T) : \sum_{t=1}^T y_t^2 x_t^\top P_t x_t \leq R \right\}.$$
Ellipsoid constraints

\[ \mathcal{Y}^T_R = \left\{ (y_1, \ldots, y_T) : \sum_{t=1}^{T} y_t^2 x_t^\top P_t x_t \leq R \right\}. \]

Minimax strategy: linear

\[ \hat{y}_n^* = x_n^\top P_n s_{n-1}. \]
Linear regression

Ellipsoid constraints

$$\mathcal{Y}_R^T = \left\{ (y_1, \ldots, y_T) : \sum_{t=1}^{T} y_t^2 x_t^\top P_t x_t \leq R \right\}.$$ 

Minimax regret $= R$.

Minimax strategy: linear

$$\hat{y}_n^* = x_n^\top P_n s_{n-1}.$$
Linear regression

Ellipsoid constraints

\[ \mathcal{Y}_R^T = \left\{ (y_1, \ldots, y_T) : \sum_{t=1}^T y_t^2 x_t^\top P_t x_t \leq R \right\}. \]

Minimax regret \( = R. \)

Minimax strategy: linear

\[ \hat{y}_n^* = x_n^\top P_n s_{n-1}. \quad (\text{MM}) \]

Equalizer property

For all \( y_1, \ldots, y_T, \)

Regret of (MM) \( := \sum_{t=1}^T (\hat{y}_t - y_t)^2 - \min_{\beta \in \mathbb{R}^p} \sum_{t=1}^T (\beta^\top x_t - y_t)^2. \)
**Linear regression**

**Ellipsoid constraints**

\[ \mathcal{Y}_R^T = \left\{ (y_1, \ldots, y_T) : \sum_{t=1}^{T} y_t^2 x_t^\top P_t x_t \leq R \right\}. \]

**Minimax regret** = \( R \).

**Minimax strategy: linear**

\[ \hat{y}_n^* = x_n^\top P_n s_{n-1}. \quad (\text{MM}) \]

**Equalizer property**

For all \( y_1, \ldots, y_T \),

\[
\text{Regret of } (\text{MM}) := \sum_{t=1}^{T} (\hat{y}_t - y_t)^2 - \min_{\beta \in \mathbb{R}^p} \sum_{t=1}^{T} \left( \beta^\top x_t - y_t \right)^2
\]

\[
= \sum_{t=1}^{T} y_t^2 x_t^\top P_t x_t.
\]
Recall:

\[ P_T^{-1} = \sum_{t=1}^{T} x_t x_t^\top, \]

\[ P_n^{-1} = \sum_{t=1}^{n} x_t x_t^\top + \sum_{t=n+1}^{T} \frac{x_t^\top P_t x_t}{1 + x_t^\top P_x x_t} x_t x_t^\top. \]

Define

\[ P_0^{-1} = \sum_{q=1}^{T} \frac{x_q^\top P_q x_q}{1 + x_q^\top P_q x_q} x_q x_q^\top \succeq 0. \]
A reformulation

$$P_0^{-1} = \sum_{q=1}^{T} \frac{x_q^\top P_q x_q}{1 + x_q^\top P_q x_q} x_q x_q^\top \succeq 0.$$ 

$$P_{t+1} = P_t - \frac{a_t}{b_t^2} P_t x_{t+1} x_{t+1}^\top P_t,$$

where

$$a_t = \frac{\sqrt{4b_t^2 + 1} - 1}{\sqrt{4b_t^2 + 1} + 1},$$

$$b_t^2 = x_{t+1}^\top P_t x_{t+1}.$$
Legal covariate sequences

For any \( t \geq 0 \), any \( x_1, \ldots, x_t \) and any \( P_t \), the following two conditions are equivalent.

1. There is a \( T \geq t \) and a sequence \( x_{t+1}, \ldots, x_T \) such that

\[
P_T^{-1} = \sum_{q=1}^{T} x_q x_q^\top.
\]

2. \( P_t^{-1} \succeq \sum_{q=1}^{t} x_q x_q^\top \).

Adversarial covariates

Thus, each \( P_0 \succeq 0 \) (a ‘covariance budget’) defines a set of sequences \( x_1, \ldots, x_T \) (and corresponding suitable bounds on \( y_1, \ldots, y_T \)). The same strategy is optimal for each of these sequences.
Linear regression

One-step constraint

Suppose we have \( P_t^{-1} \preceq \sum_{q=1}^{t} x_q x_q^\top \).

Then \( x_{t+1} \) satisfies the consistency condition

\[
P_{t+1}^{-1} \preceq \sum_{q=1}^{t+1} x_q x_q^\top,
\]

iff

1. \( x_{t+1} \) is orthogonal to the kernel of \( P_t^{-1} - \sum_{q=1}^{t} x_q x_q^\top \), and

2. \( x_{t+1}^\top P_t x_{t+1} \leq d(\hat{x}_{t+1}) + \sqrt{d(\hat{x}_{t+1})} \),

where \( \hat{x}_{t+1} = x_{t+1}/\|x_{t+1}\| \) and

\[
d(\hat{x}) = \frac{\hat{x}^\top P_t \hat{x}}{\hat{x}^\top \left( P_t + P_t \left[ \left( \sum_{q=1}^{t} x_q x_q^\top \right)^{-1} - P_t \right]^{-1} P_t \right) \hat{x}}.
\]
Minimix optimal for two families of label constraints: box constraints and problem-weighted $\ell_2$ norm constraints.
Minimax optimal for two families of label constraints: box constraints and problem-weighted $\ell_2$ norm constraints.

Strategy does not need to know the constraints.
Linear regression

\[ \hat{y}_n^* = x_n \top P_n s_{n-1} \]

- Minimax optimal for two families of label constraints: box constraints and problem-weighted $\ell_2$ norm constraints.
- Strategy does not need to know the constraints.
- Regret is $O(p \log T)$. 
Linear regression

\[ \hat{y}_n^* = x_n^\top P_n s_{n-1} \]

- Minimax optimal for two families of label constraints: box constraints and problem-weighted $\ell_2$ norm constraints.
- Strategy does not need to know the constraints.
- Regret is $O(p \log T)$.
- Same strategy is optimal for covariate sequences consistent with some ‘covariance budget’ $P_0$. 
Computing minimax optimal strategies.
Part 2: Linear regression.
Time series forecasting.
Other games with efficient minimax optimal strategies

Time series forecasting
(with Yasin Abbasi-Yadkori, Wouter Koolen, Alan Malek)

\[
\min_{a_1} \max_{x_1} \cdots \min_{a_T} \max_{x_T} \sum_{t=1}^{T} \|a_t - x_t\|^2
\]

Loss of Learner

\[
- \min_{\hat{a}_1, \ldots, \hat{a}_T} \sum_{t=1}^{T} \|\hat{a}_t - x_t\|^2 + \lambda_T \sum_{t=1}^{T+1} \|\hat{a}_t - \hat{a}_{t-1}\|^2.
\]

Loss of Comparator

Comparator Complexity

- Expression for regret when \(x_t\) bounded. (And a bound when it is not.)
- Minimax strategy makes linear predictions.
- Regret is \(\Theta \left( \frac{T}{\sqrt{1 + \lambda_T}} \right)\).
- More generally, penalize comparator by the energy of the innovations of a time series model. Efficient linear minimax strategy. Regret?

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Computing minimax optimal strategies.


Part 2: Linear regression.

Time series forecasting.