Introduction to Time Series Analysis. Lecture 7.

Peter Bartlett

Last lecture:

1. ARMA(p,q) models
2. Stationarity, causality and invertibility
3. The linear process representation of ARMA processes: ψ.

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1. Review: ARMA(p,q) models and their properties
2. Autocovariance of an ARMA process.
3. Homogeneous linear difference equations.

Review: Autoregressive moving average models

An **ARMA(p,q) process** $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q},$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

Usually, we insist that $\phi_p, \theta_q \neq 0$ and that the polynomials

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p, \qquad \theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$$

have no common factors. This implies it is not a lower order ARMA model.

Review: Properties of ARMA(p,q) models

Theorem: If ϕ and θ have no common factors, a (unique) *stationary* solution $\{X_t\}$ to $\phi(B)X_t = \theta(B)W_t$
exists iff

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0 \Rightarrow |z| \neq 1.$$

This ARMA(p,q) process is *causal* iff

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0 \Rightarrow |z| > 1.$$

It is *invertible* iff

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q = 0. \Rightarrow |z| > 1.$$

Review: Properties of ARMA(p,q) models

$$
\begin{aligned}
\phi(B)X_t = \theta(B)W_t, \quad &\Leftrightarrow \quad X_t = \psi(B)W_t \\
\text{so} \quad \theta(B) &= \psi(B)\phi(B) \\
\Leftrightarrow \; 1 + \theta_1 B + \cdots + \theta_q B^q &= (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p) \\
\Leftrightarrow \quad 1 &= \psi_0, \\
\theta_1 &= \psi_1 - \phi_1\psi_0, \\
\theta_2 &= \psi_2 - \phi_1\psi_1 - \cdots - \phi_2\psi_0, \\
&\vdots
\end{aligned}
$$

This is equivalent to $\theta_j = \phi(B)\psi_j$, with $\theta_0 = 1$, $\theta_j = 0$ for $j < 0$, $j > q$.

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1. Review: ARMA(p,q) models and their properties
2. Autocovariance of an ARMA process.
3. Homogeneous linear difference equations.

Autocovariance functions of linear processes

Consider a (mean 0) linear process $\{X_t\}$ defined by $X_t = \psi(B)W_t$.

$$
\begin{aligned}
\gamma(h) &= \mathrm{E}\left(X_t X_{t+h}\right) \\
&= \mathrm{E}\left(\psi_0 W_t + \psi_1 W_{t-1} + \psi_2 W_{t-2} + \cdots\right) \\
&\qquad \times \left(\psi_0 W_{t+h} + \psi_1 W_{t+h-1} + \psi_2 W_{t+h-2} + \cdots\right) \\
&= \sigma_w^2 \left(\psi_0 \psi_h + \psi_1 \psi_{h+1} + \psi_2 \psi_{h+2} + \cdots\right).
\end{aligned}
$$

Autocovariance functions of MA processes

Consider an MA(q) process $\{X_t\}$ defined by $X_t = \theta(B)W_t$.

$$\gamma(h) = \begin{cases} \sigma_w^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h} & \text{if } h \leq q, \\ 0 & \text{if } h > q. \end{cases}$$

Autocovariance functions of ARMA processes

ARMA process: $\phi(B)X_t = \theta(B)W_t$.

To compute γ, we can compute ψ, and then use

$$\gamma(h) = \sigma_w^2 \left(\psi_0\psi_h + \psi_1\psi_{h+1} + \psi_2\psi_{h+2} + \cdots\right).$$

Autocovariance functions of ARMA processes

An alternative approach:

$$\begin{aligned}
X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}& \\
&= W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q}, \\
\text{so } \mathrm{E}\left(\left(X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}\right) X_{t-h}\right)& \\
&= \mathrm{E}\left(\left(W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q}\right) X_{t-h}\right), \\
\text{that is, } \gamma(h) - \phi_1 \gamma(h-1) - \cdots - \phi_p \gamma(h-p)& \\
&= \mathrm{E}\left(\theta_h W_{t-h} X_{t-h} + \cdots + \theta_q W_{t-q} X_{t-h}\right) \\
&= \sigma_w^2 \sum_{j=0}^{q-h} \theta_{h+j} \psi_j. \qquad \text{(Write } \theta_0 = 1\text{).}
\end{aligned}$$

This is a linear difference equation.

Autocovariance functions of ARMA processes: Example

$$(1 + 0.25B^2)X_t = (1 + 0.2B)W_t, \qquad \Leftrightarrow \qquad X_t = \psi(B)W_t,$$

$$\psi_j = \left(1, \frac{1}{5}, -\frac{1}{4}, -\frac{1}{20}, \frac{1}{16}, \frac{1}{80}, -\frac{1}{64}, -\frac{1}{320}, \ldots\right).$$

$$\gamma(h) - \phi_1\gamma(h-1) - \phi_2\gamma(h-2) = \sigma_w^2 \sum_{j=0}^{q-h} \theta_{h+j}\psi_j$$

$$\Leftrightarrow \gamma(h) + 0.25\gamma(h-2) = \begin{cases} \sigma_w^2 (\psi_0 + 0.2\psi_1) & \text{if } h = 0, \\ 0.2\sigma_w^2 \psi_0 & \text{if } h = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Autocovariance functions of ARMA processes: Example

We have the homogeneous linear difference equation

$$\gamma(h) + 0.25\gamma(h-2) = 0$$

for $h \geq 2$, with initial conditions

$$\gamma(0) + 0.25\gamma(-2) = \sigma_w^2\,(1 + 1/25)$$
$$\gamma(1) + 0.25\gamma(-1) = \sigma_w^2/5.$$

We can solve these linear equations to determine γ.

Or we can use the theory of linear difference equations...

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Difference equations

Examples:

$$\begin{aligned} x_t - 3x_{t-1} &= 0 \qquad \text{(first order, linear)} \\ x_t - x_{t-1}x_{t-2} &= 0 \qquad \text{(3rd order, nonlinear)} \\ x_t + 2x_{t-1} - x_{t-3}^2 &= 0 \qquad \text{(3rd order, nonlinear)} \end{aligned}$$

Homogeneous linear diff eqns with constant coefficients

$$
\begin{aligned}
a_0 x_t + a_1 x_{t-1} + \cdots + a_k x_{t-k} &= 0 \\
\Leftrightarrow \qquad \left(a_0 + a_1 B + \cdots + a_k B^k\right) x_t &= 0 \\
\Leftrightarrow \qquad a(B) x_t &= 0 \\
\text{auxiliary equation:} \qquad a_0 + a_1 z + \cdots + a_k z^k &= 0 \\
\Leftrightarrow \qquad (z - z_1)(z - z_2) \cdots (z - z_k) &= 0
\end{aligned}
$$

where $z_1, z_2, \ldots, z_k \in \mathbb{C}$ are the roots of this *characteristic polynomial*.

Thus,

$$a(B)x_t = 0 \qquad \Leftrightarrow \qquad (B - z_1)(B - z_2) \cdots (B - z_k)x_t = 0.$$

Homogeneous linear diff eqns with constant coefficients

$$a(B)x_t = 0 \qquad \Leftrightarrow \qquad (B - z_1)(B - z_2) \cdots (B - z_k)x_t = 0.$$

So any $\{x_t\}$ satisfying $(B - z_i)x_t = 0$ for some i also satisfies $a(B)x_t = 0$.

Three cases:

1. The z_i are real and distinct.
2. The z_i are complex and distinct.
3. Some z_i are repeated.

Homogeneous linear diff eqns with constant coefficients

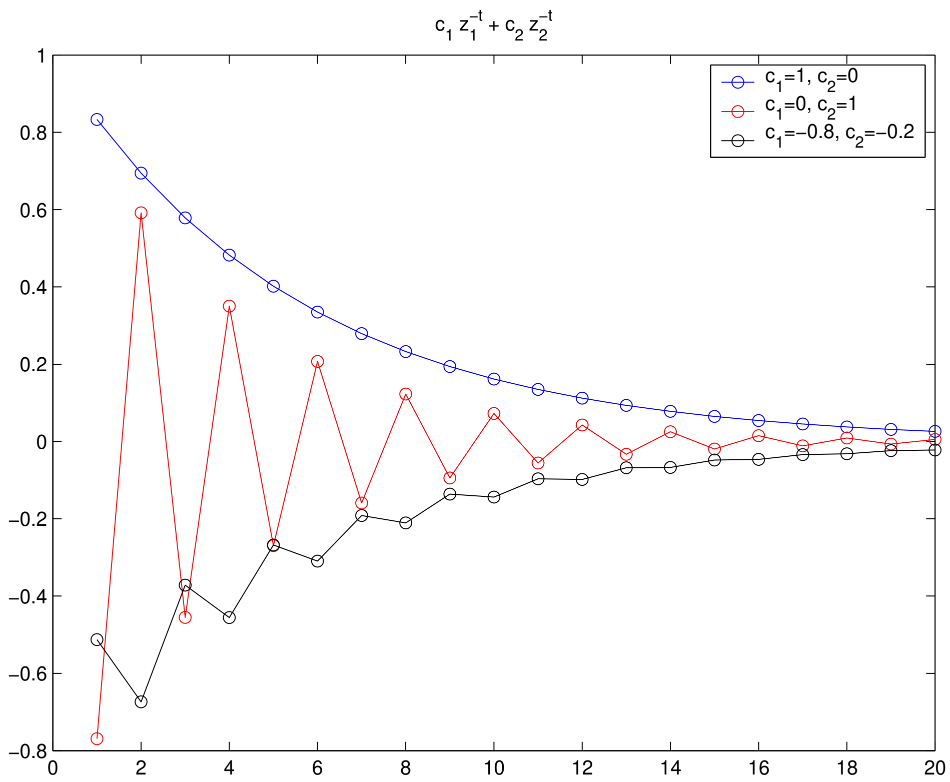
1. The z_i are real and distinct.

$$
\begin{aligned}
& & a(B)x_t &= 0 \\
&\Leftrightarrow & (B - z_1)(B - z_2)\cdots(B - z_k)x_t &= 0 \\
&\Leftrightarrow & x_t \text{ is a linear combination of solutions to} & \\
& & (B - z_1)x_t = 0,\ (B - z_2)x_t = 0, \ldots,\ (B - z_k)x_t &= 0 \\
&\Leftrightarrow & x_t = c_1 z_1^{-t} + c_2 z_2^{-t} + \cdots + c_k z_k^{-t}&,
\end{aligned}
$$

for some constants $c_1, \ldots, c_k$.

Homogeneous linear diff eqns with constant coefficients

1. The z_i are real and distinct. $z_1 = 1.2, z_2 = -1.3$



Complex exponentials

$$a + ib = re^{i\theta} = r(\cos\theta + i\sin\theta),$$

$$\text{where} \quad r = |a + ib| = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \in (-\pi, \pi].$$

Thus, $$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1 + \theta_2)},$$

$$z\bar{z} = |z|^2.$$

Homogeneous linear diff eqns with constant coefficients

2. The z_i are complex and distinct.

As before, $$a(B)x_t = 0$$

$$\Leftrightarrow \qquad x_t = c_1 z_1^{-t} + c_2 z_2^{-t} + \cdots + c_k z_k^{-t}.$$

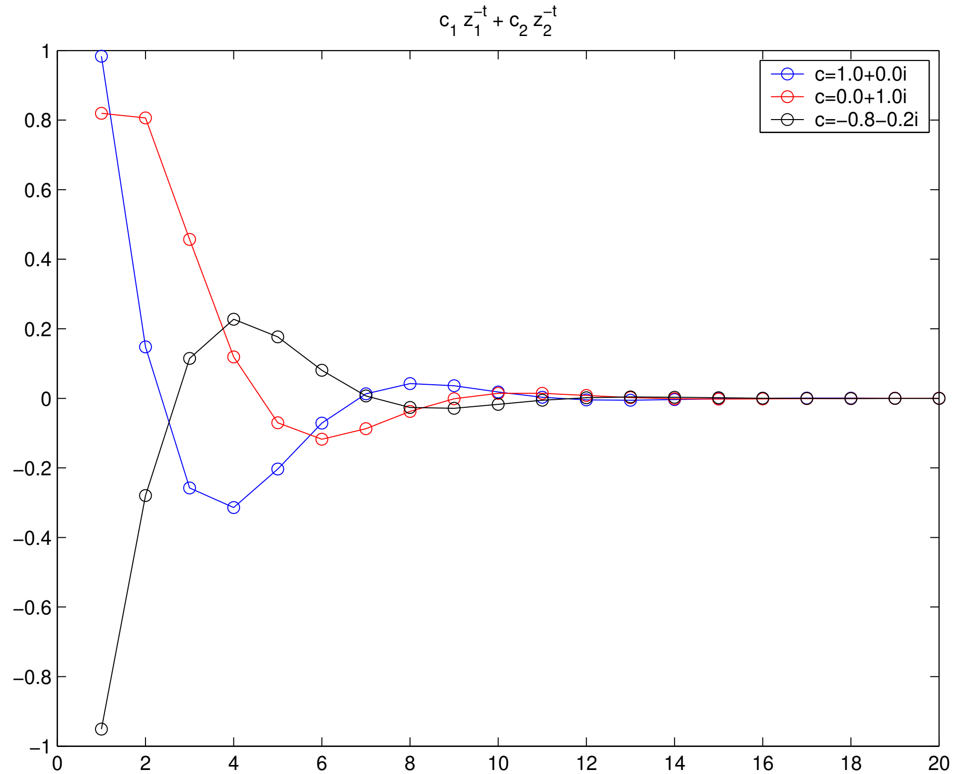
If $z_1 \notin \mathbb{R}$, since $a_1, \ldots, a_k$ are real, we must have the complex conjugate root, $z_j = \bar{z_1}$. And for x_t to be real, we must have $c_j = \bar{c_1}$. For example:

$$\begin{aligned} x_t &= c\, z_1^{-t} + \bar{c}\, \bar{z_1}^{-t} \\ &= r\, e^{i\theta} |z_1|^{-t} e^{-i\omega t} + r\, e^{-i\theta} |z_1|^{-t} e^{i\omega t} \\ &= r|z_1|^{-t} \left(e^{i(\theta - \omega t)} + e^{-i(\theta - \omega t)} \right) \\ &= 2r|z_1|^{-t} \cos(\omega t - \theta) \end{aligned}$$

where $z_1 = |z_1| e^{i\omega}$ and $c = r e^{i\theta}$.

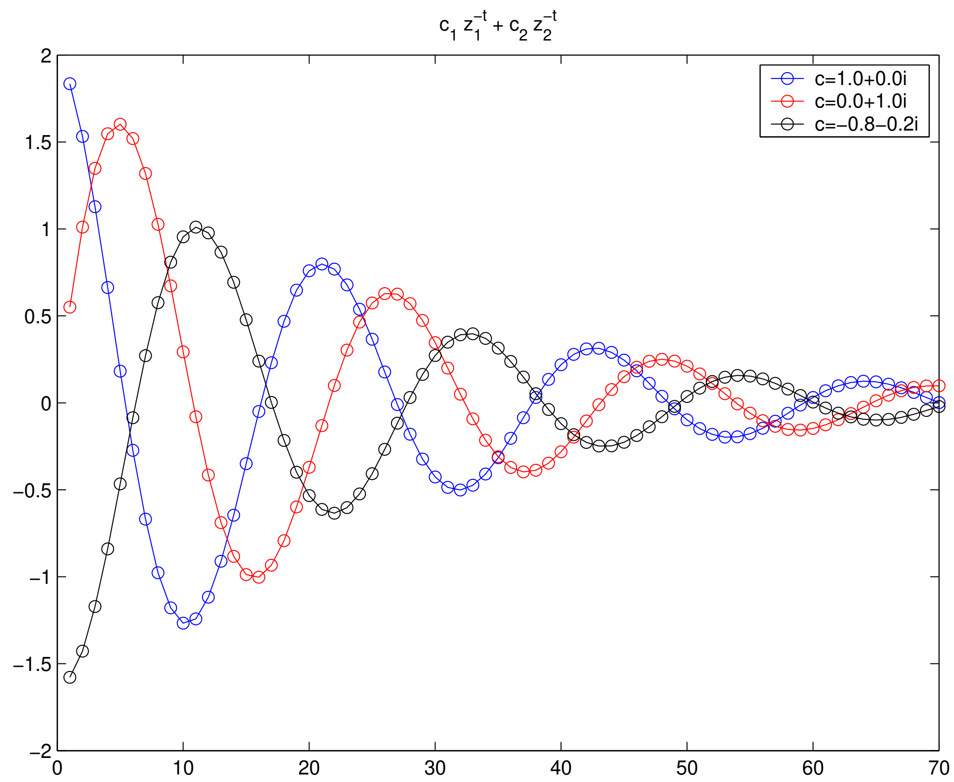
Homogeneous linear diff eqns with constant coefficients

2. The z_i are complex and distinct. $z_1 = 1.2 + i$, $z_2 = 1.2 - i$



Homogeneous linear diff eqns with constant coefficients

2. The z_i are complex and distinct. $z_1 = 1 + 0.1i$, $z_2 = 1 - 0.1i$



Homogeneous linear diff eqns with constant coefficients

3. Some z_i are repeated.

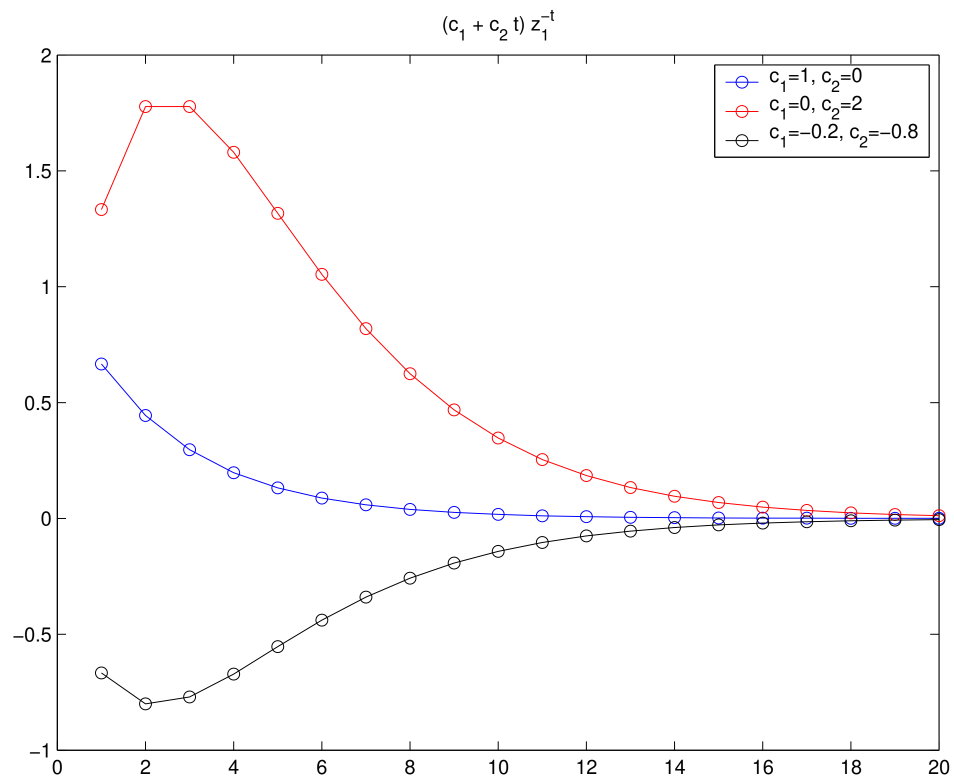
$$
\begin{aligned}
& & a(B)x_t &= 0 \\
&\Leftrightarrow & (B-z_1)(B-z_2)\cdots(B-z_k)x_t &= 0. \\
\text{If } z_1 = z_2, & & (B-z_1)(B-z_2)x_t &= 0 \\
&\Leftrightarrow & (B-z_1)^2 x_t &= 0.
\end{aligned}
$$

We can check that $(c_1 + c_2 t) z_1^{-t}$ is a solution...

More generally, $(B - z_1)^m x_t = 0$ has the solution $\left(c_1 + c_2 t + \cdots + c_{m-1} t^{m-1}\right) z_1^{-t}$.

Homogeneous linear diff eqns with constant coefficients

3. Some z_i are repeated. $z_1 = z_2 = 1.5.$



Solving linear diff eqns with constant coefficients

$$a_0 x_t + a_1 x_{t-1} + \cdots + a_k x_{t-k} = 0,$$

$$\text{with initial conditions} \quad x_1, \ldots, x_k.$$

Auxiliary equation in $z \in \mathbb{C}$: $\qquad a_0 + a_1 z + \cdots + a_k z^k = 0$

$$\Leftrightarrow \qquad (z - z_1)^{m_1} (z - z_2)^{m_2} \cdots (z - z_l)^{m_l} = 0,$$

where $z_1, z_2, \ldots, z_l \in \mathbb{C}$ are the roots of the characteristic polynomial, and z_i occurs with multiplicity m_i.

Solutions: $c_1(t) z_1^{-t} + c_2(t) z_2^{-t} + \cdots + c_l(t) z_l^{-t}$,

where $c_i(t)$ is a polynomial in t of degree $m_i - 1$.

We determine the coefficients of the $c_i(t)$ using the initial conditions (which might be linear constraints on the initial values $x_1, \ldots, x_k$).

Autocovariance functions of ARMA processes: Example

$$(1 + 0.25B^2)X_t = (1 + 0.2B)W_t, \qquad \Leftrightarrow \qquad X_t = \psi(B)W_t,$$

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$$\gamma(h) - \phi_1\gamma(h-1) - \phi_2\gamma(h-2) = \sigma_w^2 \sum_{j=0}^{q-h} \theta_{h+j}\psi_j$$

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Autocovariance functions of ARMA processes: Example

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Autocovariance functions of ARMA processes: Example

Homogeneous lin. diff. eqn:

$$\gamma(h) + 0.25\gamma(h-2) = 0.$$

The characteristic polynomial is

$$1 + 0.25z^2 = \frac{1}{4}\left(4 + z^2\right) = \frac{1}{4}(z-2i)(z+2i),$$

which has roots at $z_1 = 2e^{i\pi/2}$, $\bar{z_1} = 2e^{-i\pi/2}$.
The solution is of the form

$$\gamma(h) = cz_1^{-h} + \bar{c}\bar{z_1}^{-h}.$$

Autocovariance functions of ARMA processes: Example

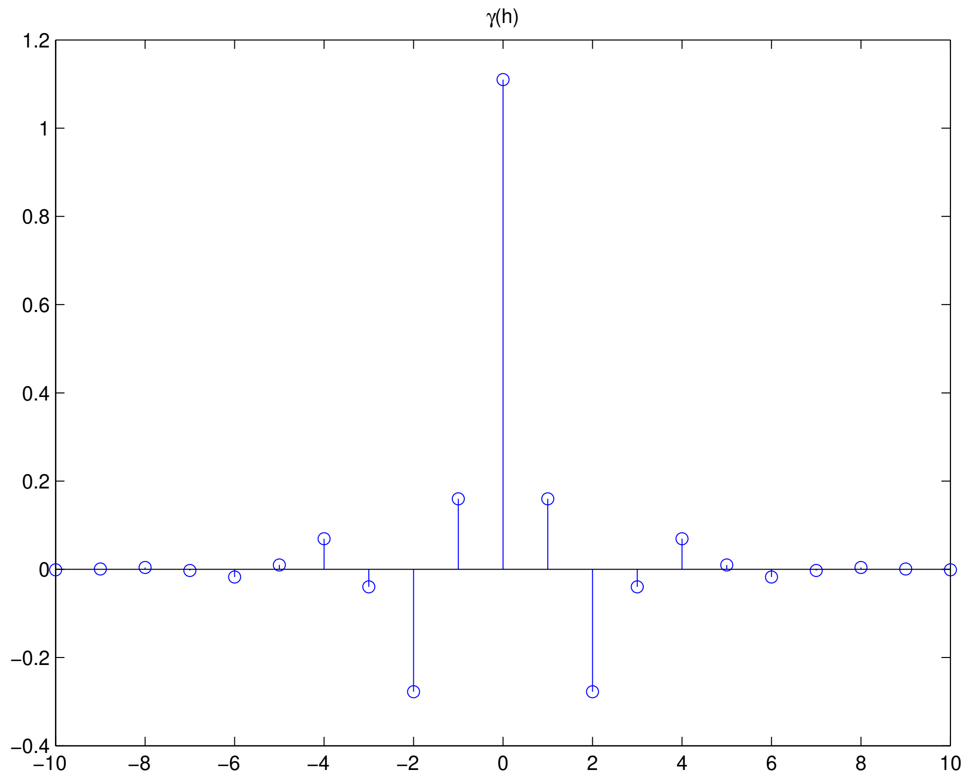
$z_1 = 2e^{i\pi/2}, \bar{z_1} = 2e^{-i\pi/2}, c = |c|e^{i\theta}.$

We have
$$\begin{aligned}\gamma(h) &= c z_1^{-h} + \bar{c}\bar{z_1}^{-h} \\ &= 2^{-h}\left(|c|e^{i(\theta-h\pi/2)} + |c|e^{i(-\theta+h\pi/2)}\right) \\ &= c_1 2^{-h}\cos\left(\frac{h\pi}{2} - \theta\right).\end{aligned}$$

And we determine c_1, θ from the initial conditions

$$\begin{aligned}\gamma(0) + 0.25\gamma(-2) &= \sigma_w^2\left(1 + 1/25\right) \\ \gamma(1) + 0.25\gamma(-1) &= \sigma_w^2/5.\end{aligned}$$

Autocovariance functions of ARMA processes: Example



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