Introduction to Time Series Analysis. Lecture 5.

Peter Bartlett

Last lecture:

1. ACF, sample ACF
2. Properties of the sample ACF
3. Convergence in mean square

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1. AR(1) as a linear process
2. Causality
3. Invertibility
4. AR(p) models

AR(1) as a linear process

Let $\{X_t\}$ be the stationary solution to $X_t - \phi X_{t-1} = W_t$, where $W_t \sim WN(0, \sigma^2)$.

If $|\phi| < 1$,

$$X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}$$

is the unique solution:

- This infinite sum converges in mean square, since $|\phi| < 1$ implies $\sum |\phi^j| < \infty$.
- It satisfies the AR(1) recurrence.

AR(1) in terms of the back-shift operator

We can write

$$
\begin{aligned}
& & X_t - \phi X_{t-1} &= W_t \\
\Leftrightarrow & & \underbrace{(1 - \phi B)}_{\phi(B)} X_t &= W_t \\
\Leftrightarrow & & \phi(B) X_t &= W_t
\end{aligned}
$$

Recall that B is the back-shift operator: $BX_t = X_{t-1}$.

AR(1) in terms of the back-shift operator

Also, we can write

$$
\begin{aligned}
& & X_t &= \sum_{j=0}^{\infty} \phi^j W_{t-j} \\
&\Leftrightarrow & X_t &= \underbrace{\sum_{j=0}^{\infty} \phi^j B^j}_{\pi(B)} W_t \\
&\Leftrightarrow & X_t &= \pi(B) W_t
\end{aligned}
$$

AR(1) in terms of the back-shift operator

With these definitions:

$$\pi(B) = \sum_{j=0}^{\infty} \phi^j B^j \qquad \text{and} \qquad \phi(B) = 1 - \phi B,$$

we can check that $\pi(B) = \phi(B)^{-1}$:

$$\pi(B)\phi(B) = \sum_{j=0}^{\infty} \phi^j B^j (1 - \phi B) = \sum_{j=0}^{\infty} \phi^j B^j - \sum_{j=1}^{\infty} \phi^j B^j = 1.$$

$$\begin{aligned} \text{Thus,} \qquad & \phi(B) X_t = W_t \\ \Rightarrow \qquad & \pi(B)\phi(B) X_t = \pi(B) W_t \\ \Leftrightarrow \qquad & X_t = \pi(B) W_t. \end{aligned}$$

AR(1) in terms of the back-shift operator

Notice that manipulating operators like $\phi(B)$, $\pi(B)$ is like manipulating polynomials:

$$\frac{1}{1-\phi z} = 1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \cdots ,$$

provided $|\phi| < 1$ and $|z| \leq 1$.

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AR(1) and Causality

Let X_t be the stationary solution to

$$X_t - \phi X_{t-1} = W_t,$$

where $W_t \sim WN(0, \sigma^2)$.

If $|\phi| < 1$,

$$X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}.$$

$\phi = 1$?

$\phi = -1$?

$|\phi| > 1$?

AR(1) and Causality

If $|\phi| > 1$, $\pi(B)W_t$ does not converge.
But we can rearrange

$$X_t = \phi X_{t-1} + W_t$$

$$\text{as} \qquad X_{t-1} = \frac{1}{\phi}X_t - \frac{1}{\phi}W_t,$$

and we can check that the unique stationary solution is

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} W_{t+j}.$$

But... X_t depends on **future** values of W_t.

Causality

A linear process $\{X_t\}$ is **causal** (strictly, a **causal function of** $\{W_t\}$) if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

with $$\sum_{j=0}^{\infty} |\psi_j| < \infty$$

and $$X_t = \psi(B) W_t.$$

AR(1) and Causality

- Causality is a property of $\{X_t\}$ **and** $\{W_t\}$.
- Consider the AR(1) process defined by $\phi(B)X_t = W_t$ (with $\phi(B) = 1 - \phi B$):

$$
\begin{array}{ll}
 & \phi(B)X_t = W_t \quad \text{is causal} \\
\text{iff} & |\phi| < 1 \\
\text{iff} & \text{the root } z_1 \text{ of the polynomial } \phi(z) = 1 - \phi z \text{ satisfies } |z_1| > 1.
\end{array}
$$

AR(1) and Causality

- Consider the AR(1) process $\phi(B)X_t = W_t$ (with $\phi(B) = 1 - \phi B$): If $|\phi| > 1$, we can define an equivalent causal model,

$$X_t - \phi^{-1}X_{t-1} = \tilde{W}_t,$$

where $\tilde{W}_t$ is a new white noise sequence.

AR(1) and Causality

- Is an MA(1) process causal?

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MA(1) and Invertibility

Define

$$\begin{aligned} X_t &= W_t + \theta W_{t-1} \\ &= (1 + \theta B) W_t. \end{aligned}$$

If $|\theta| < 1$, we can write

$$\begin{aligned} & (1+\theta B)^{-1} X_t = W_t \\ \Leftrightarrow \qquad & (1 - \theta B + \theta^2 B^2 - \theta^3 B^3 + \cdots) X_t = W_t \\ \Leftrightarrow \qquad & \sum_{j=0}^{\infty} (-\theta)^j X_{t-j} = W_t. \end{aligned}$$

That is, we can write W_t as a *causal* function of X_t.
We say that this MA(1) is **invertible**.

MA(1) and Invertibility

$$X_t = W_t + \theta W_{t-1}$$

If $|\theta| > 1$, the sum $\sum_{j=0}^{\infty} (-\theta)^j X_{t-j}$ diverges, but we can write

$$W_{t-1} = -\theta^{-1} W_t + \theta^{-1} X_t.$$

Just like the noncausal AR(1), we can show that

$$W_t = -\sum_{j=1}^{\infty} (-\theta)^{-j} X_{t+j}.$$

That is, we can write W_t as a linear function of X_t, but it is not causal. We say that this MA(1) is not **invertible**.

Invertibility

A linear process $\{X_t\}$ is **invertible** (strictly, an **invertible function of** $\{W_t\}$) if there is a

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\pi_j| < \infty$$

$$\text{and} \qquad W_t = \pi(B) X_t.$$

MA(1) and Invertibility

- Invertibility is a property of $\{X_t\}$ **and** $\{W_t\}$.
- Consider the MA(1) process defined by $\phi(B)X_t = W_t$ (with $\phi(B) = 1 - \phi B$):

$$\begin{array}{ll} & X_t = \theta(B)W_t \quad \text{is invertible} \\ \text{iff} & |\theta| < 1 \\ \text{iff} & \text{the root } z_1 \text{ of the polynomial } \theta(z) = 1 + \theta z \text{ satisfies } |z_1| > 1. \end{array}$$

MA(1) and Invertibility

- Consider the MA(1) process $X_t = \theta(B)W_t$ (with $\theta(B) = 1 + \theta B$): If $|\theta| > 1$, we can define an equivalent invertible model in terms of a new white noise sequence.
- Is an AR(1) process invertible?

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AR(p): Autoregressive models of order p

An **AR(p) process** $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t,$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

Equivalently, $\qquad \phi(B) X_t = W_t,$

where $\qquad \phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p.$

AR(p): Constraints on ϕ

Recall: For $p = 1$ (AR(1)), $\phi(B) = 1 - \phi_1 B$.

This is an AR(1) model only if there is a *stationary* solution to $\phi(B)X_t = W_t$, which is equivalent to $|\phi_1| \neq 1$.

This is equivalent to the following condition on $\phi(z) = 1 - \phi_1 z$:

$$\forall z \in \mathbb{R},\ \phi(z) = 0 \ \Rightarrow\ z \neq \pm 1$$

$$\text{equivalently, } \forall z \in \mathbb{C},\ \phi(z) = 0 \ \Rightarrow\ |z| \neq 1,$$

where $\mathbb{C}$ is the set of complex numbers.

AR(p): Constraints on ϕ

$$\text{Stationarity:} \qquad \forall z \in \mathbb{C},\ \phi(z) = 0 \ \Rightarrow\ |z| \neq 1,$$

where $\mathbb{C}$ is the set of complex numbers.

$\phi(z) = 1 - \phi_1 z$ has one root at $z_1 = 1/\phi_1 \in \mathbb{R}$.
But the roots of a degree $p > 1$ polynomial might be complex.
For stationarity, we want the roots of $\phi(z)$ to avoid the **unit circle**,
$\{z \in \mathbb{C} : |z| = 1\}$.

AR(p): Stationarity and causality

Theorem: A (unique) *stationary* solution to $\phi(B)X_t = W_t$ exists iff

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0 \Rightarrow |z| \neq 1.$$

This AR(p) process is *causal* iff

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0 \Rightarrow |z| > 1.$$

Recall: Causality

A linear process $\{X_t\}$ is **causal** (strictly, a **causal function of** $\{W_t\}$) if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty$$

$$\text{and} \qquad X_t = \psi(B) W_t.$$

AR(p): Roots outside the unit circle implies causal (Details)

$$\forall z \in \mathbb{C},\ |z| \leq 1 \ \Rightarrow\ \phi(z) \neq 0$$

$$\Leftrightarrow \qquad \exists \{\psi_j\}, \delta > 0,\ \forall |z| \leq 1 + \delta,\ \frac{1}{\phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j.$$

$$\Rightarrow \qquad \forall |z| \leq 1 + \delta,\ |\psi_j z^j| \to 0,\ \left(|\psi_j|^{1/j} |z|\right)^j \to 0$$

$$\Rightarrow \qquad \exists j_0,\ \forall j \geq j_0,\ |\psi_j|^{1/j} \leq \frac{1}{1 + \delta/2} \qquad \Rightarrow \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty.$$

So if $|z| \leq 1 \Rightarrow \phi(z) \neq 0$, then $S_m = \sum_{j=0}^{m} \psi_j B^j W_t$ converges in mean square, so we have a stationary, causal time series $X_t = \phi^{-1}(B) W_t$.

Calculating ψ for an AR(p): matching coefficients

Example: $X_t = \psi(B)W_t \quad \Leftrightarrow \quad (1 - 0.5B + 0.6B^2)X_t = W_t,$

$$
\begin{aligned}
\text{so} \quad 1 &= \psi(B)(1 - 0.5B + 0.6B^2) \\
\Leftrightarrow \quad 1 &= (\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots)(1 - 0.5B + 0.6B^2) \\
\Leftrightarrow \quad 1 &= \psi_0, \\
0 &= \psi_1 - 0.5\psi_0, \\
0 &= \psi_2 - 0.5\psi_1 + 0.6\psi_0, \\
0 &= \psi_3 - 0.5\psi_2 + 0.6\psi_1, \\
&\vdots
\end{aligned}
$$

Calculating ψ for an AR(p): example

$$
\begin{aligned}
\Leftrightarrow \qquad & 1 = \psi_0, \qquad 0 = \psi_j \qquad (j \le 0), \\
& 0 = \psi_j - 0.5\psi_{j-1} + 0.6\psi_{j-2} \\
\Leftrightarrow \qquad & 1 = \psi_0, \qquad 0 = \psi_j \qquad (j \le 0), \\
& 0 = \phi(B)\psi_j.
\end{aligned}
$$

We can solve these *linear difference equations* in several ways:

- numerically, or
- by guessing the form of a solution and using an inductive proof, or
- by using the theory of linear difference equations.

Calculating ψ for an AR(p): general case

$$\phi(B)X_t = W_t, \qquad \Leftrightarrow \qquad X_t = \psi(B)W_t$$

$$\text{so} \qquad 1 = \psi(B)\phi(B)$$

$$\Leftrightarrow \qquad 1 = (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p)$$

$$\begin{aligned}
\Leftrightarrow \qquad 1 &= \psi_0, \\
0 &= \psi_1 - \phi_1 \psi_0, \\
0 &= \psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0, \\
&\vdots \\
\Leftrightarrow \qquad 1 &= \psi_0, \qquad 0 = \psi_j \qquad (j < 0), \\
0 &= \phi(B)\psi_j.
\end{aligned}$$

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