Introduction to Time Series Analysis. Lecture 5. Peter Bartlett

Last lecture:

- 1. ACF, sample ACF
- 2. Properties of the sample ACF
- 3. Convergence in mean square

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- 1. AR(1) as a linear process
- 2. Causality
- 3. Invertibility
- 4. AR(p) models

AR(1) as a linear process

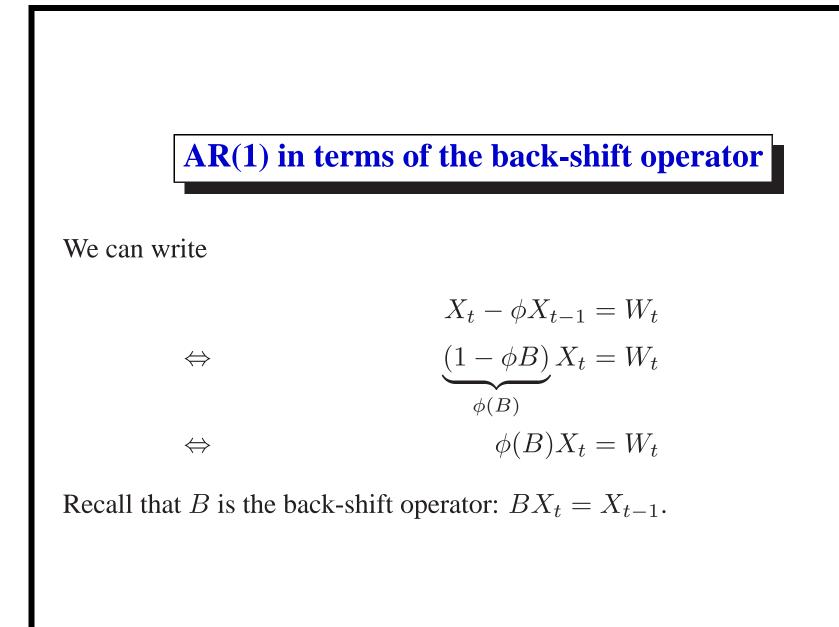
Let $\{X_t\}$ be the stationary solution to $X_t - \phi X_{t-1} = W_t$, where $W_t \sim WN(0, \sigma^2).$ If $|\phi| < 1$,

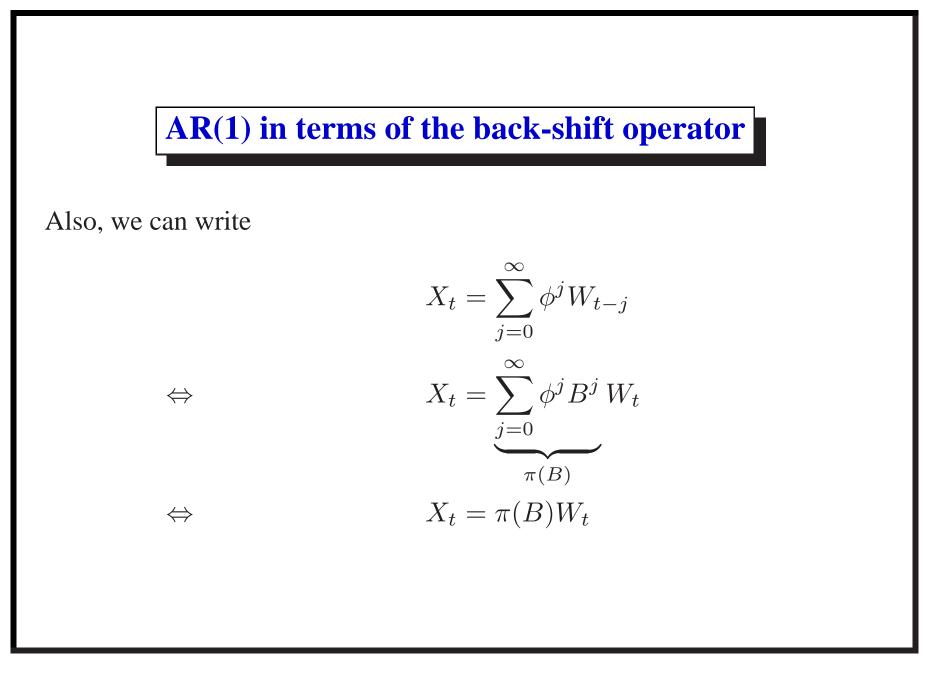
$$X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}$$

is the unique solution:

• This infinite sum converges in mean square, since $|\phi| < 1$ implies $\sum |\phi^j| < \infty.$

• It satisfies the AR(1) recurrence.





AR(1) in terms of the back-shift operator

With these definitions:

$$\pi(B) = \sum_{j=0}^{\infty} \phi^j B^j \quad \text{and} \quad \phi(B) = 1 - \phi B,$$

we can check that $\pi(B) = \phi(B)^{-1}$:

$$\pi(B)\phi(B) = \sum_{j=0}^{\infty} \phi^{j}B^{j}(1-\phi B) = \sum_{j=0}^{\infty} \phi^{j}B^{j} - \sum_{j=1}^{\infty} \phi^{j}B^{j} = 1.$$

Thus, $\phi(B)X_{t} = W_{t}$
 $\Rightarrow \quad \pi(B)\phi(B)X_{t} = \pi(B)W_{t}$
 $\Leftrightarrow \quad X_{t} = \pi(B)W_{t}.$

AR(1) in terms of the back-shift operator

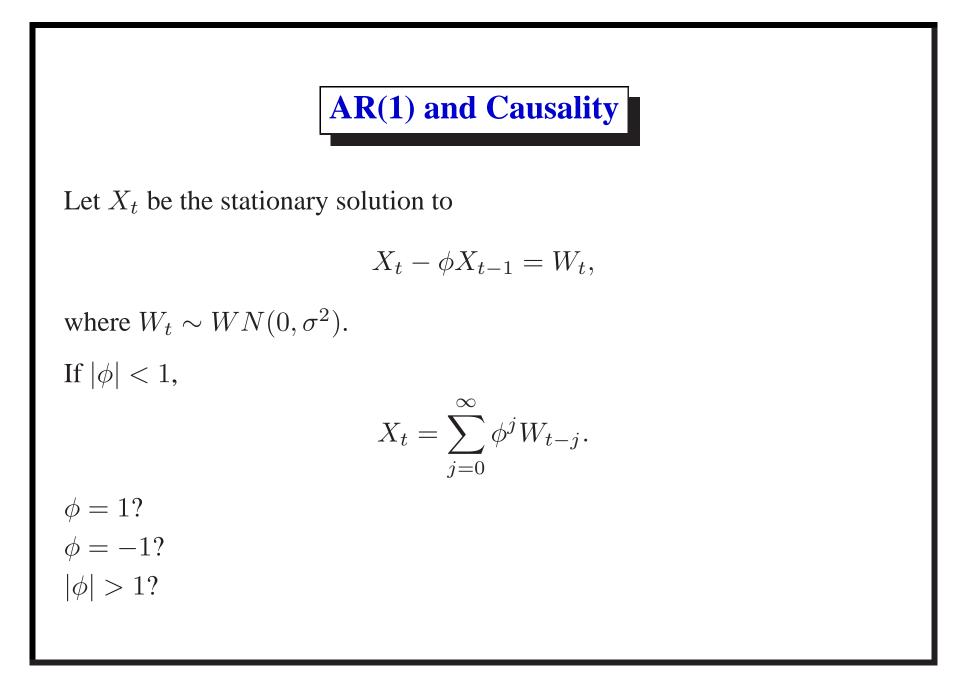
Notice that manipulating operators like $\phi(B)$, $\pi(B)$ is like manipulating polynomials:

$$\frac{1}{1-\phi z} = 1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \cdots,$$

provided $|\phi| < 1$ and $|z| \leq 1$.

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AR(1) and Causality

If $|\phi| > 1$, $\pi(B)W_t$ does not converge. But we can rearrange

$$X_t = \phi X_{t-1} + W_t$$

as
$$X_{t-1} = \frac{1}{\phi} X_t - \frac{1}{\phi} W_t,$$

and we can check that the unique stationary solution is

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} W_{t+j}.$$

But... X_t depends on **future** values of W_t .

Causality

A linear process $\{X_t\}$ is **causal** (strictly, a **causal function** of $\{W_t\}$) if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

with
$$\sum_{j=0}^{\infty} |\psi_j| < \infty$$

and $X_t = \psi(B)W_t$.

AR(1) and Causality

- Causality is a property of $\{X_t\}$ and $\{W_t\}$.
- Consider the AR(1) process defined by $\phi(B)X_t = W_t$ (with $\phi(B) = 1 \phi B$):

 $\phi(B)X_t = W_t$ is causal

 $\quad \text{iff} \qquad |\phi| < 1$

iff the root z_1 of the polynomial $\phi(z) = 1 - \phi z$ satisfies $|z_1| > 1$.

AR(1) and Causality

 Consider the AR(1) process φ(B)X_t = W_t (with φ(B) = 1 - φB): If |φ| > 1, we can define an equivalent causal model,

$$X_t - \phi^{-1} X_{t-1} = \tilde{W}_t,$$

where \tilde{W}_t is a new white noise sequence.



• Is an MA(1) process causal?

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MA(1) and Invertibility

Define

$$X_t = W_t + \theta W_{t-1}$$
$$= (1 + \theta B) W_t.$$

If $|\theta| < 1$, we can write

$$(1 + \theta B)^{-1} X_t = W_t$$

$$\Leftrightarrow \qquad (1 - \theta B + \theta^2 B^2 - \theta^3 B^3 + \cdots) X_t = W_t$$

$$\Leftrightarrow \qquad \sum_{j=0}^{\infty} (-\theta)^j X_{t-j} = W_t.$$

That is, we can write W_t as a *causal* function of X_t . We say that this MA(1) is **invertible**. MA(1) and Invertibility

$$X_t = W_t + \theta W_{t-1}$$

If $|\theta| > 1$, the sum $\sum_{j=0}^{\infty} (-\theta)^j X_{t-j}$ diverges, but we can write

$$W_{t-1} = -\theta^{-1}W_t + \theta^{-1}X_t.$$

Just like the noncausal AR(1), we can show that

$$W_t = -\sum_{j=1}^{\infty} (-\theta)^{-j} X_{t+j}.$$

That is, we can write W_t as a linear function of X_t , but it is not causal. We say that this MA(1) is not **invertible**.

Invertibility

A linear process $\{X_t\}$ is **invertible** (strictly, an **invertible function of** $\{W_t\}$) if there is a

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots$$

with
$$\sum_{j=0}^{\infty} |\pi_j| < \infty$$

and $W_t = \pi(B)X_t$.

MA(1) and Invertibility

- Invertibility is a property of $\{X_t\}$ and $\{W_t\}$.
- Consider the MA(1) process defined by $\phi(B)X_t = W_t$ (with $\phi(B) = 1 \phi B$):

 $X_t = \theta(B)W_t$ is invertible

iff $|\theta| < 1$

iff the root z_1 of the polynomial $\theta(z) = 1 + \theta z$ satisfies $|z_1| > 1$.

MA(1) and Invertibility

- Consider the MA(1) process X_t = θ(B)W_t (with θ(B) = 1 + θB): If |θ| > 1, we can define an equivalent invertible model in terms of a new white noise sequence.
- Is an AR(1) process invertible?

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AR(p): Autoregressive models of order *p*

An **AR**(**p**) process $\{X_t\}$ is a stationary process that satisfies $X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = W_t$, where $\{W_t\} \sim WN(0, \sigma^2)$.

Equivalently,
$$\phi(B)X_t = W_t$$
,
where $\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$.

AR(p): Constraints on ϕ

Recall: For p = 1 (AR(1)), $\phi(B) = 1 - \phi_1 B$. This is an AR(1) model only if there is a *stationary* solution to $\phi(B)X_t = W_t$, which is equivalent to $|\phi_1| \neq 1$. This is equivalent to the following condition on $\phi(z) = 1 - \phi_1 z$:

 $\forall z \in \mathbb{R}, \ \phi(z) = 0 \Rightarrow z \neq \pm 1$

equivalently, $\forall z \in \mathbb{C}, \ \phi(z) = 0 \implies |z| \neq 1$,

where \mathbb{C} is the set of complex numbers.

AR(p): Constraints on ϕ

Stationarity: $\forall z \in \mathbb{C}, \ \phi(z) = 0 \implies |z| \neq 1,$

where \mathbb{C} is the set of complex numbers.

 $\phi(z) = 1 - \phi_1 z$ has one root at $z_1 = 1/\phi_1 \in \mathbb{R}$. But the roots of a degree p > 1 polynomial might be complex. For stationarity, we want the roots of $\phi(z)$ to avoid the **unit circle**, $\{z \in \mathbb{C} : |z| = 1\}.$

AR(p): Stationarity and causality

Theorem: A (unique) *stationary* solution to $\phi(B)X_t = W_t$ exists iff

$$\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p = 0 \implies |z| \neq 1.$$

This AR(p) process is *causal* iff

$$\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p = 0 \implies |z| > 1.$$

Recall: Causality

A linear process $\{X_t\}$ is **causal** (strictly, a **causal function** of $\{W_t\}$) if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

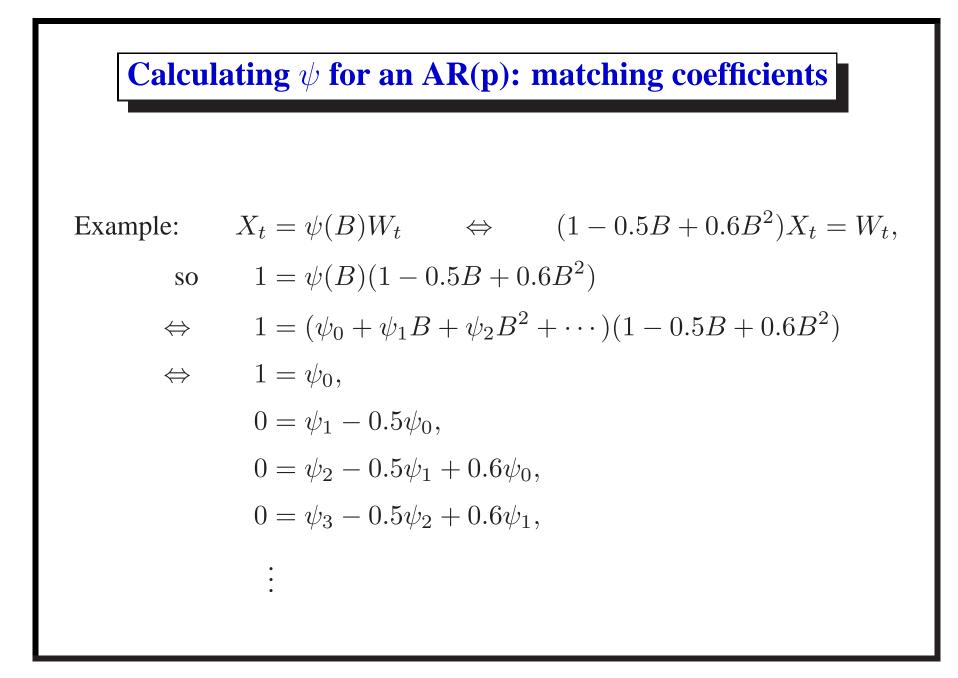
with
$$\sum_{j=0}^{\infty} |\psi_j| < \infty$$

and $X_t = \psi(B)W_t$.

AR(p): Roots outside the unit circle implies causal (Details)

$$\begin{aligned} \forall z \in \mathbb{C}, \ |z| \leq 1 \Rightarrow \phi(z) \neq 0 \\ \Leftrightarrow \qquad \exists \{\psi_j\}, \delta > 0, \ \forall |z| \leq 1 + \delta, \ \frac{1}{\phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j. \\ \Rightarrow \qquad \forall |z| \leq 1 + \delta, \ |\psi_j z^j| \to 0, \ \left(|\psi_j|^{1/j} |z|\right)^j \to 0 \\ \Rightarrow \qquad \exists j_0, \ \forall j \geq j_0, \ |\psi_j|^{1/j} \leq \frac{1}{1 + \delta/2} \qquad \Rightarrow \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty. \end{aligned}$$

So if
$$|z| \leq 1 \Rightarrow \phi(z) \neq 0$$
, then $S_m = \sum_{j=0} \psi_j B^j W_t$ converges in mean square, so we have a stationary, causal time series $X_t = \phi^{-1}(B)W_t$.



Calculating ψ **for an AR(p): example**

$$\Rightarrow \qquad 1 = \psi_0, \qquad 0 = \psi_j \qquad (j \le 0), \\ 0 = \psi_j - 0.5\psi_{j-1} + 0.6\psi_{j-2} \\ \Leftrightarrow \qquad 1 = \psi_0, \qquad 0 = \psi_j \qquad (j \le 0), \\ 0 = \phi(B)\psi_j.$$

We can solve these *linear difference equations* in several ways:

- numerically, or
- by guessing the form of a solution and using an inductive proof, or
- by using the theory of linear difference equations.

Calculating ψ for an AR(p): general case

$$\begin{split} \phi(B)X_t &= W_t, \quad \Leftrightarrow \quad X_t = \psi(B)W_t \\ \text{so} & 1 = \psi(B)\phi(B) \\ \Leftrightarrow & 1 = (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p) \\ \Leftrightarrow & 1 = \psi_0, \\ 0 &= \psi_1 - \phi_1\psi_0, \\ 0 &= \psi_2 - \phi_1\psi_1 - \phi_2\psi_0, \\ \vdots \\ \Leftrightarrow & 1 = \psi_0, \quad 0 = \psi_j \quad (j < 0), \\ 0 &= \phi(B)\psi_j. \end{split}$$

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